Gabor Lens Theory and Mathematica Results

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1 Equations of Motion

Based on the setup of the Gabor lens, an electron plasma can be confined in the lens. A magnetic field which lies along the longitudinal direction (i.e. $B\hat{\boldsymbol{z}}$) confines the plasma transversely, while the cylindrical electrode system creates a potential well confining the plasma along the longitudinal direction.

The trajectory of an electron entering such a setup can be determined by a balance of forces given by the Lorentz force:

$$m_e \frac{d\boldsymbol{v}}{dt} = -e\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) \tag{1}$$

where m_e is the electron mass, $\boldsymbol{v}$ is the electron velocity, e is the magnitude of the electron charge, $\boldsymbol{E}$ is the electric field, and $\boldsymbol{B}$ is the magnetic field. Writing out the components we get:

$$\begin{aligned} m_e \tfrac{dv_x}{dt} &= -e \times \left(E_x + (v_y B_z - v_z B_y)\right) \\ m_e \tfrac{dv_y}{dt} &= -e \times \left(E_y - (v_x B_z - v_z B_x)\right) \\ m_e \tfrac{dv_z}{dt} &= -e \times \left(E_z + (v_x B_y - v_y B_x)\right) \end{aligned}$$

We can rewrite this using the Newtonian dot notation, (i.e. $\dot{x} = \frac{dx}{dt} = v_x$):

$$\begin{aligned} m_e \ddot{x} &= -e \times \left(E_x + (\dot{y} B_z - \dot{z} B_y)\right) \\ m_e \ddot{y} &= -e \times \left(E_y - (\dot{x} B_z - \dot{z} B_x)\right) \\ m_e \ddot{z} &= -e \times \left(E_z + (\dot{x} B_y - \dot{y} B_x)\right) \end{aligned} \tag{2}$$

It is due to the coupling between the system of equations that makes finding an analytical solution extremely complicated. Using *Mathematica* we can try to determine a trajectory either analytically (if possible) or numerically. However, we first need to ensure that the results obtained match with those given by the theory which assumed cylindrical symmetry before playing around with these equation. The main unknown in this equation is the electric field which comes about as a property of the plasma and the anode. Rather than attempt to explain everything in terms of Cartesian coordinates where an analytical solution is extremely complex, it is easiest to work in cylindrical coordinates. I'll explain in Section 6 a more 'general' way to do this in Cartesian coordinates, which is the approach the code aims to achieve.

2 Cylindrical Symmetry

2.1 Assumptions

Based on the Gabor lens setup, when we assume cylindrical symmetry we can impose the assumptions of applying a homogeneous magnetic field symmetrically along the longitudinal direction and for the plasma to only affect particles in the radial direction:

$$\begin{aligned} \text{Radial Electric Field :} \quad & E = E_r(r) \\ \text{Axially-Symmetric Magnetic Field :} \quad & B = B_z \end{aligned}$$

Recall that the cylindrical coordinate system are given as:

$$\begin{aligned} x =& r\cos(\phi) \\ y =& r\sin(\phi) \\ z =& z \end{aligned}$$

Hence, this essentially means that we can set $B_x = B_y = 0$, and $E_z = 0$:

$$\begin{array}{rc} m_e\ddot{x} = & -e \times (E_x + \dot{y}B_z) \\ m_e\ddot{y} = & -e \times (E_y - \dot{x}B_z) \\ m_e\ddot{z} = & 0 \end{array} \tag{3}$$

2.2 Electric Field

We wish to determine the electric field. To do so, we assume a cold and homogeneously distributed electron plasma in the cylindrical region with no neutralizing ion background. We proceed by making use of Gauss's Law to get the electric space charge field. The electric field in an infinite cylinder of uniform charge points radially outwards with an electric flux Φ:

$$\Phi = E(2\pi rL) = \frac{Q'}{\epsilon_0}$$

where we assume a cylinder of length L, with radius r, enclosed charge by the Gaussian surface of Q', and vacuum permittivity ϵ_0. A Gaussian surface encloses less than the total charge, which we relate to the total charge by:

$$\frac{Q'}{Q} = \frac{\pi r^2 L}{\pi R^2 L} \qquad \Rightarrow \qquad Q' = Q\frac{r^2}{R^2}$$

with R being the radius from the center of cylinder to the surface. We can substitute this result into our flux to get:

$$\Phi = E(2\pi rL) = \frac{Qr^2}{\epsilon_0 R^2} \qquad \Rightarrow \qquad E = \frac{Qr}{\epsilon_0 2\pi R^2 L}$$

Since the volume of the cylinder is given by $\pi R^2 L$, we can express the electric field E in terms of the electron number density n_e by letting $\frac{Q}{\pi R^2 L} = -en_e$:

$$E = -\frac{n_e e}{2\epsilon_0} r = E_r \tag{4}$$

This is in terms of the radial coordinate, if we want to express this in Cartesian coordinates then we consider:

$$\hat{\boldsymbol{r}} = \cos(\phi)\hat{\boldsymbol{x}} + \sin(\phi)\hat{\boldsymbol{y}} \qquad \text{and} \qquad \phi = \arcsin\left(\frac{y}{r}\right) = \arcsin\left(\frac{y}{\sqrt{x^2+y^2}}\right) \tag{5}$$

Hence,

$$\begin{aligned}
E_r\hat{\boldsymbol{r}} =& E_r\left(\cos(\phi)\hat{\boldsymbol{x}} + \sin(\phi)\hat{\boldsymbol{y}}\right) \\
=& E_r\left(\cos\left[\arcsin\left(\frac{y}{r}\right)\right]\hat{\boldsymbol{x}} + \sin\left[\arcsin\left(\frac{y}{r}\right)\right]\hat{\boldsymbol{y}}\right) \\
=& E_r\sqrt{1-\frac{y^2}{x^2+y^2}}\hat{\boldsymbol{x}} + E_r\frac{y}{r}\hat{\boldsymbol{y}} \\
=& E_r\left(\frac{x}{r}\hat{\boldsymbol{x}} + \frac{y}{r}\hat{\boldsymbol{y}}\right)
\end{aligned}$$

This shows that in terms of the Cartesian coordinates, the radial electric field is expressed as:

$$E_x = -\frac{n_e e}{2\epsilon_0}x \tag{6}$$

$$E_y = -\frac{n_e e}{2\epsilon_0}y \tag{7}$$

To summarize, by making some assumptions about the plasma we determined the electric field radially, and converted it to Cartesian units. Hence, we can update our system of equations:

$$\begin{aligned}
m_e\ddot{x} =& \quad -e\times\left(-\tfrac{n_e e}{2\epsilon_0}x + \dot{y}B_z\right) \\
m_e\ddot{y} =& \quad -e\times\left(-\tfrac{n_e e}{2\epsilon_0}y - \dot{x}B_z\right) \\
m_e\ddot{z} =& \quad 0
\end{aligned} \tag{8}$$

We have most of what we need, what remains is to determine the electron density for the confinement of the electron plasma in the Gabor lens. One thing to note here is that for the equation of motion for the z-axis, we see that it says that electrons would stream out of the lends longitudinally (because we stated $B_x = B_y = E_z = 0$). As mentioned at the start, we confine the plasma longitudinally by adding the ground electrodes which create a potential well to trap the electron plasma. Thus, the longitudinal equation of motion will need to be modified to include this. This will be discussed in Sec. 4.

3 Radial Confinement

We now wish to determine the maximum electron density that can be confined in the radial direction. As the plasma column rotates about the axis of symmetry of the lens, different forces will act on the electron. In the radial direction, the balance of forces is given by:

$$-\frac{m_e {v_\phi}^2}{r} = -eE_r - ev_\phi B_z \tag{9}$$

Recalling the argument from above, the radial electric field is given by Eq. 4, re-expressed as:

$$E_r = -\frac{n_e e}{2\epsilon_0} r = -\frac{m_e}{2e} {\omega_p}^2 r$$

where we used the definition for the plasma frequency as $\omega_p = \sqrt{\frac{n_e e^2}{\epsilon_0 m_e}}$. In addition, we will also introduce the angular velocity defined as $\omega_e = \frac{v_\phi}{r}$ such that we rewrite Eq. 9 as:

$$\begin{aligned} -m_e v_\phi \omega_e &= \frac{m_e}{2} {\omega_p}^2 r - e\omega_e r B_z \\ {\omega_e}^2 &= -\frac{{\omega_p}^2}{2} + \omega_e \Omega_e \end{aligned}$$

where we used the definition of the electron cyclotron frequency $\Omega_e = \frac{eB_z}{m_e}$. Next applying the quadratic formula will give the solution for ω_e as:

$$\omega_e^{\pm} = \frac{\Omega_e}{2}\left[1 \pm \sqrt{1 - \frac{2{\omega_p}^2}{{\Omega_e}^2}}\right] \tag{10}$$

This result states that there are 2 possible mean rotation velocities for the plasma column. The Brillouin flow limit is the high density limit where these two solutions coalesce. As can be easily seen, this occurs when: $2{\omega_p}^2 = {\Omega_e}^2$, such that $\omega_e = \frac{\Omega_e}{2}$. This refers to a rigid rotation of the plasma column with an angular velocity $\frac{\Omega_e}{2}$.

A reminder is that we want to find the maximal electron density such that the plasma is still confined radially. So we need to relate this result to the electron number density. We have found the condition on ω_p for the Brillouin flow limit, hence:

$$\begin{aligned} {\omega_p}^2 &= \frac{n_e e^2}{\epsilon_0 m_e} \\ \frac{{\Omega_e}^2}{2} &= \frac{n_e e^2}{\epsilon_0 m_e} \\ \frac{e^2 {B_z}^2}{2{m_e}^2} &= \frac{n_e e^2}{\epsilon_0 m_e} \\ \boxed{n_{e,r,\max} = \frac{\epsilon_0 {B_z}^2}{2m_e}} \end{aligned} \tag{11}$$

3.1 Visualization

It is useful to be able to visualize the Brillouin flow limit and how it will affect the trajectory of an electron. The equations of motion were solved and plotted with *Mathematica*. Fig. 1 shows the trajectory for an electron starting at the middle of the lens $(0, 0, \frac{l}{2})$ (defining $z = 0$ to be at the entrance of the lens with length $\ell = 500 \times 10^{-3}$ m) with a velocity of 1 eV in the x-direction, at $0.9 \times$ "Brillouin Flow Limit", and a magnetic field of $B_z = 0.008$ T. As can be seen, the electron appears to be radially confined, tracing out a circular region. Fig. 2 shows an electron with the same parameters as in Fig. 1 but beginning at an initial position of $(0.007\text{ m}, 0, \frac{\ell}{2})$. As can be seen, the electron traces out an annular-like region, never reaching a centre region. Fig. 3 has the electron beginning at $(0, 0, \frac{\ell}{2})$ but with velocity instead in the y-direction at 20 eV. This slightly affected the trajectory at the start but at later time steps stills fills out a circular area, albeit, encircling a larger region.

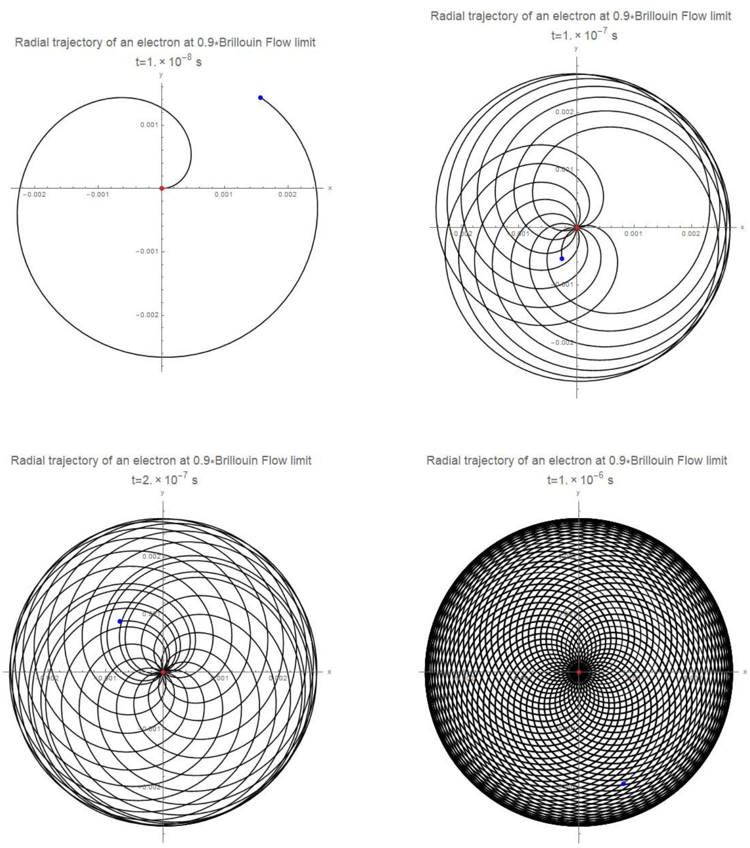

Figure 1: Parametric plots of the radial trajectory of an electron initially at $(0, 0, \frac{\ell}{2})$ with velocity of $(1\text{ eV}, 0, 0)$ encountering a plasma at an electron density of $0.9 \times$ Brillouin Flow Limit, and a longitudinal magnetic field $B_z = 0.008$ T. The red point represents the initial position of electron at $t = 0$ and the blue point shows the electron at the final position at a given time t.

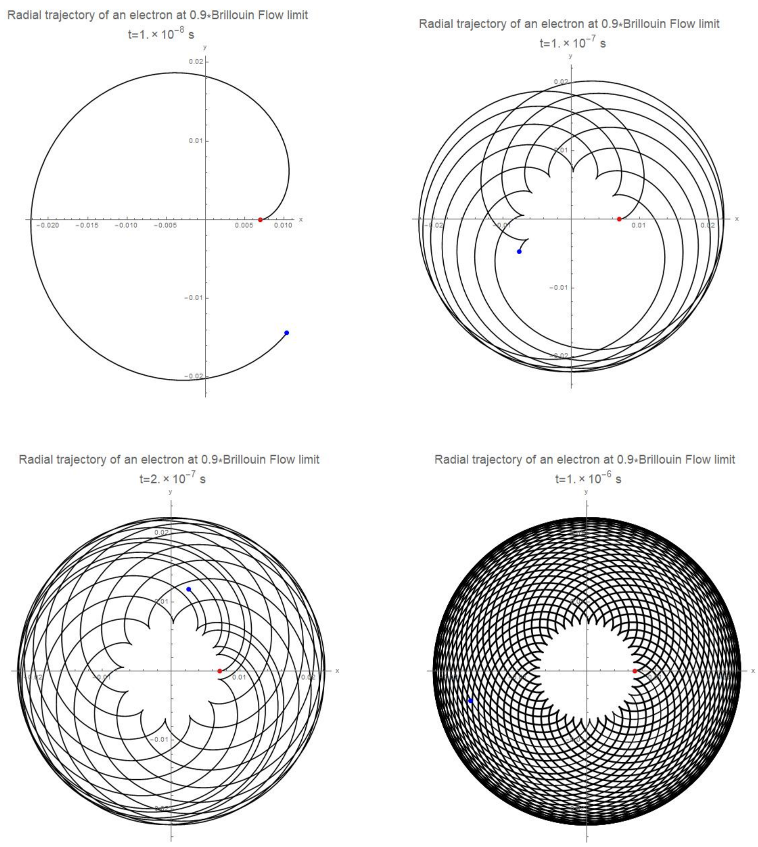

Figure 2: Parametric plots of the radial trajectory of an electron initially at $(0.007 \text{ m}, 0, \frac{\ell}{2})$ with velocity of $(1 \text{ eV}, 0, 0)$ encountering a plasma at an electron density of $0.9 \times$ Brillouin Flow Limit, and a longitudinal magnetic field $B_z = 0.008$ T. The red point represents the initial position of electron at $t = 0$ and the blue point shows the electron at the final position at a given time t.

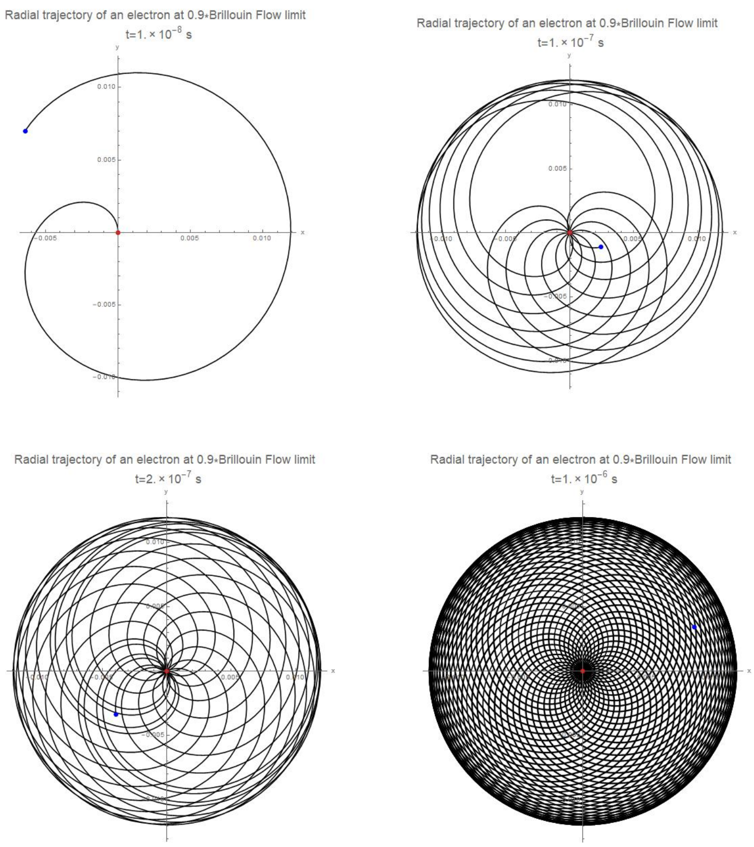

Figure 3: Parametric plots of the radial trajectory of an electron initially at $(0, 0, \frac{\ell}{2})$ with velocity of $(0, 20\ \text{eV}, 0)$ encountering a plasma at an electron density of $0.9 \times$ Brillouin Flow Limit, and a longitudinal magnetic field $B_z = 0.008$ T. The red point represents the initial position of electron at $t = 0$ and the blue point shows the electron at the final position at a given time t.

Now, if we look at the electron trajectory when we are at about the Brillouin flow limit as given by Fig. 4 with an electron with the same parameters as in Fig. 1 except for the electron density, we see that the electron no longer traces out a set area. Instead, the trajectory indicates that the electron spirals outwards, indicating that it is no longer being confined radially. If we assume some lens parameters, such as an anode radius of 35×10^{-3} m and a length of 500×10^{-3} m, the final plot at $t = 5.89716 \times 10^{-8}$ is the point at which the electron touches the lens at which point the code stops. It should be mentioned the slight gap between the blue point and the trajectory is due to a *Mathematica* plotting issue, and can be ignored.

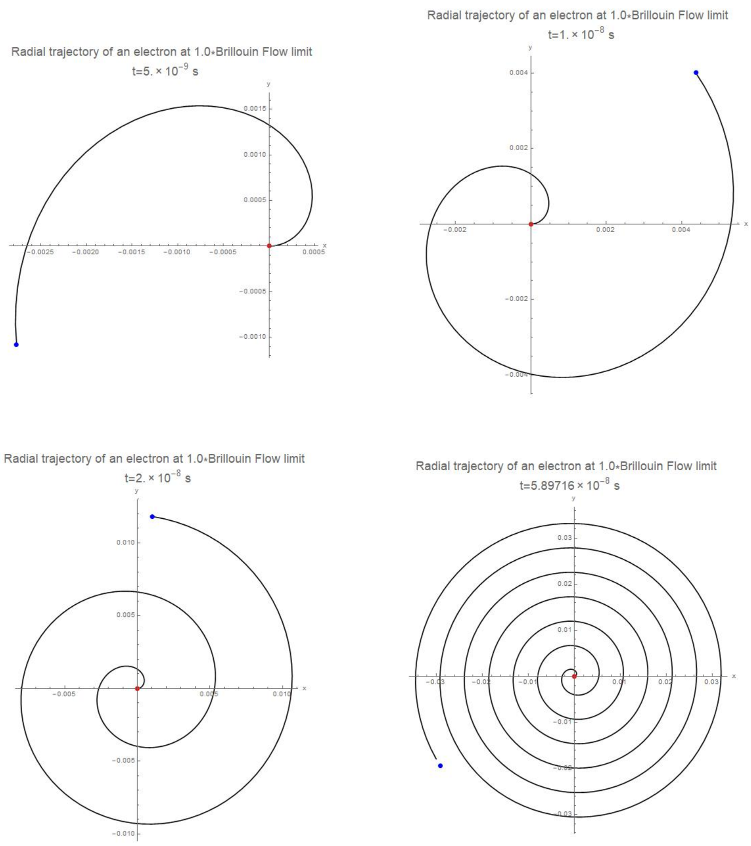

Figure 4: Parametric plots of the radial trajectory of an electron initially at $(0, 0, \frac{\ell}{2})$ with velocity of $(1\ \text{eV}, 0, 0)$ encountering a plasma at an electron density at the Brillouin Flow Limit, and a longitudinal magnetic field $B_z = 0.008$ T. The red point represents the initial position of electron at $t = 0$ and the blue point shows the electron at the final position at a given time t.

Finally, Fig. 5 shows an electron with the same parameters as Fig. 1, except we at an electron density of $1.1 \times$ "Brillouin Flow Limit". The electron quickly spirals outwards and is completely lost, i.e. is not confined radially.

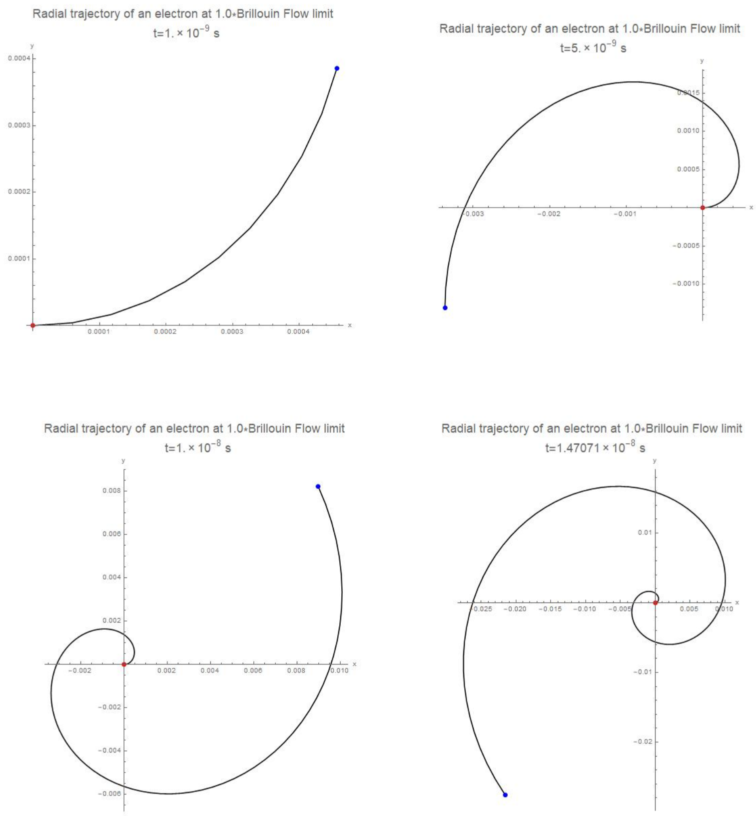

Figure 5: Parametric plots of the radial trajectory of an electron initially at $(0, 0, \frac{\ell}{2})$ with velocity of $(1 \text{ eV}, 0, 0)$ encountering a plasma at an electron density of $1.1 \times$ Brillouin Flow Limit, and a longitudinal magnetic field $B_z = 0.008$ T. The red point represents the initial position of electron at $t = 0$ and the blue point shows the electron at the final position at a given time t.

4 Longitudinal Confinement

As stated earlier, the set of differential equations given by Eq. 8 does not contain any longitudinal confining conditions. Electrons would just stream through the lens if not for the potential well setup by the cylindrical electrode system.

However, the confined electron plasma will decrease the positive potential of the anode. We can obtain the maximum confinement when the space charge potential of the electron plasma matches that of the anode potential. If we ignore the endcap electrodes constraining the maximal radius of the plasma we can obtain a simple expression for the maximum longitudinal electron number density.

4.1 'Ideal' Plasma Potential

As usual, we will consider a cold, homogeneously distributed plasma column. The potential of the electron plasma is given by:

$$V_p(r) = -\int E(r)dr = -\int \left(-\frac{n_e e}{2\epsilon_0} r\right) dr = \frac{n_e e}{4\epsilon_0} r^2$$

Evaluating this integral from 0 to the boundary, i.e. the radius R_A of the anode then:

$$V_p(R_A) = \frac{n_e e {R_A}^2}{4\epsilon_0}$$

We get the maximum confinement when the electron plasma equals the anode potential at $r = R_A$, such that the potential drops to zero and electrons will begin to no longer be confined, i.e. ($V_p(R_A) = V_A$):

$$\boxed{n_{e,l,\max} = \frac{4\epsilon_0 V_A}{e {R_A}^2}} \tag{12}$$

This gives the maximal confinement electron density condition for the longitudinal direction, but we need to determine the potential of the anode. As a side note, the overall potential would then be when we subtract from the potential of the anode the potential of the plasma (i.e. $V = V_A - V_p$).

4.2 Anode Potential - Analytical Solution

To obtain the anode potential, it requires solving Laplace's equation. We will assume the anode in the Gabor lens to be a hollow right circular cylinder. We will let it have a radius R_A and have its axis on the z-axis to range from $z = 0$ to $z = \ell$. The potential on the end faces will be assumed to be zero, corresponding to the ground electrodes, while the potential on the cylindrical surface is given to be a constant V_0.

Boundary Conditions

Reiterating the boundary conditions in mathematical notation, with the potential given by V:

$$V(r, \phi, 0) = 0 \tag{13}$$

$$V(r, \phi, \ell) = 0 \tag{14}$$

$$V(R_A, \phi, z) = V_0 \tag{15}$$

Laplace's Equation in Cylindrical Coordinates

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{16}$$

The standard approach is to use the separation of variables technique to find the solution, such that we assume $V(r,\phi,z) = R(r)\Phi(\phi)Z(z)$. Hence proceeding from Eq. 16,

$$\frac{\Phi(\phi)Z(z)}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R(r)}{\partial r}\right) + \frac{R(r)Z(z)}{r^2}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} + R(r)\Phi(\phi)\frac{\partial^2 Z(z)}{\partial z^2} = 0$$
$$\frac{1}{rR(r)}\frac{\partial R(r)}{\partial r} + \frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r^2\Phi(\phi)}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} + \frac{1}{Z(z)}\frac{\partial^2 Z(z)}{\partial z^2} = 0$$

By bearing in mind that $\phi = \phi + 2\pi$ and contemplating the solution that satisfies this, we obtain $\Phi(\phi)$ through solving:

$$\frac{1}{r^2}\frac{\partial^2 \Phi(\phi)}{\partial \phi^2} = -\frac{\nu^2}{r^2}\Phi(\phi) \tag{17}$$

where ν is an integer. The general solution is given by:

$$\boxed{\Phi(\phi) = A\sin(\nu\phi) + B\cos(\nu\phi)} \tag{18}$$

where A and B are constants. Next due to considerations of the boundary conditions of Eq. 13 and Eq. 14 which we need to satisfy, we obtain $Z(z)$ through solving:

$$\frac{\partial^2 Z(z)}{\partial z^2} = -k^2 Z(z) \tag{19}$$

where k is an integer. The general solution is given by:

$$\boxed{Z(z) = C\sin(kz) + D\cos(kz)} \tag{20}$$

where C and D are constants. As a result of what we imposed in Eq. 17 and Eq. 19, the radial portion is given by:

$$\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}\frac{\partial R(r)}{\partial r} = \left(k^2 + \frac{\nu^2}{r^2}\right) \tag{21}$$

The general solution is hence given by:

$$\boxed{R(r) = EI_\nu(kr) + FK_\nu(kr)} \tag{22}$$

where E and F are constants and $I_\nu(kr)$ is the modified Bessel function of the first kind, and $K_\nu(kr)$ is the modified Bessel function of the second kind. So in general, the solution of Laplace's equation in cylindrical coordinates would be a superposition of terms of the form:

$$\begin{aligned} V(r,\phi,z) =& R(r)\Phi(\phi)Z(z) \\ =& \sum_{\nu=0}^{\infty}\left[E_\nu I_\nu(kr) + F_\nu K_\nu(kr)\right]\left[A_\nu\sin(\nu\phi) + B_\nu\cos(\nu\phi)\right]\left[C_\nu\sin(kz) + D_\nu\cos(kz)\right] \end{aligned}$$

We will now impose the boundary conditions into this general equation. From Eq. 13 where $V(r,\phi,0)=0$ we must have $D_\nu=0$. Further, Eq. 14 will imply that:

$$Z(\ell)=C\sin(k\ell)=0$$

for this to be satisfied we would require:

$$k\ell=n\pi \qquad \Rightarrow \qquad k_n=\frac{n\pi}{\ell}, \quad n=1,2,3,\ldots$$

Next the plots giving the modified Bessel functions are given below in Fig. 6:

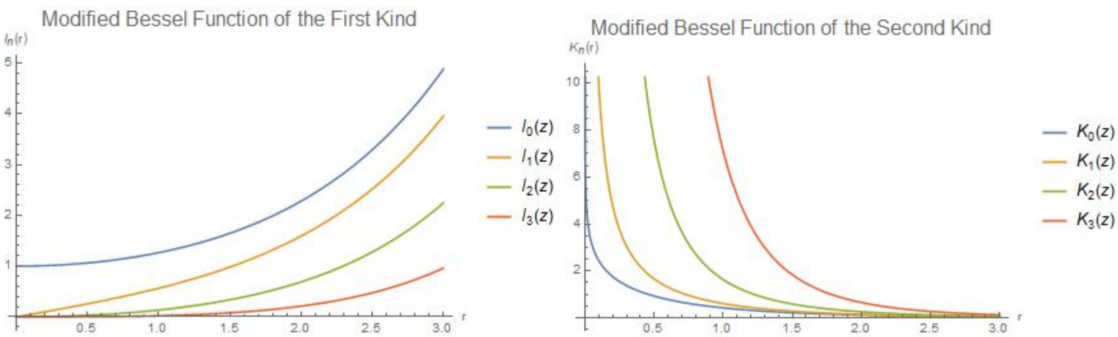

Figure 6: Plots of the modified Bessel functions of the first kind (left) and of the second kind(right).

For the cylindrical anode there is no charge at the origin, such that we expect a finite solution when $r\to 0$. By looking at the modified Bessel functions, we must necessarily impose that $F_\nu=0$. Hence, we have simplified the solution to the form below where we will also combine the integration constants:

$$V(r,\phi,z)=\sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty}\left[A_{n\nu}\sin(\nu\phi)+B_{n\nu}\cos(\nu\phi)\right]\sin(k_nz)I_\nu(k_nr)$$

We next impose the last boundary condition Eq. 15:

$$V(R_A,\phi,z)=V_0=\sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty}\left[A_{n\nu}\sin(\nu\phi)+B_{n\nu}\cos(\nu\phi)\right]\sin(k_nz)I_\nu(k_nR_A)$$

We will multiply both sides of the equation by $\sin(k_{n'}z)$ and integrate with respect to z.

$$\begin{aligned}\int_0^L V_0\sin(k_{n'}z)dz &= \sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty}\left[A_{n\nu}\sin(\nu\phi)+B_{n\nu}\cos(\nu\phi)\right]I_\nu(k_nR_A)\int_0^L\sin(k_nz)\sin(k_{n'}z)dz\\ -V_0\frac{\cos(k_{n'}z)}{k_{n'}}\bigg|_0^L &= \sum_{n=1}^{\infty}\sum_{\nu=0}^{\infty}\left[A_{n\nu}\sin(\nu\phi)+B_{n\nu}\cos(\nu\phi)\right]I_\nu(k_nR_A)\left(\frac{L}{2}\delta_{nn'}\right)\\ -\frac{V_0\ell}{n'\pi}\left[\cos(n'\pi)-1\right] &= \frac{\ell}{2}\sum_{n'=1}^{\infty}\sum_{\nu=0}^{\infty}\left[A_{n'\nu}\sin(\nu\phi)+B_{n'\nu}\cos(\nu\phi)\right]I_\nu(k_{n'}R_A)\end{aligned}$$

We make note that $\cos(n'\pi) = (-1)^{n'}$, such that for even n' we get zero. Hence, by relabeling $n = n'$ and for odd n:

$$\frac{2V_0\ell}{n\pi} = \frac{\ell}{2} \sum_{n=\text{odd}}^{\infty} \sum_{\nu=0}^{\infty} \left[A_{n\nu} \sin(\nu\phi) + B_{n\nu} \cos(\nu\phi)\right] I_\nu(k_n R_A)$$

Finally we note how the LHS contains no $\sin(\nu\phi)$ or $\cos(\nu\phi)$ terms. This would imply that $\nu = 0$, which means $A_{n\nu} = 0$. Which then means:

$$B_{n0} = \frac{4V_0}{n\pi} \frac{1}{I_0(k_n R_A)}$$

We finally have the solution:

$$\boxed{V(r, \phi, z) = \frac{4V_0}{\pi} \sum_{n=\text{odd}}^{\infty} \frac{1}{n} \frac{\sin(k_n z) I_0(k_n r)}{I_0(k_n R_A)}} \tag{23}$$

To visually see this solution we will input some test values. We will let $V_0 = 630$ V, $R_A = 0.035$ m, $\ell = 0.5$ m. In addition, the sum is to infinity, but we can't realistically do such an infinite sum except for certain values of the radius, For illustrative purposes we will sum 50 odd n terms. This is given in Fig. 7, where in the plot on the right we see some oscillations near the edges, which is a result of not summing to infinity. In theory it should be a perfect straight line with an immediate drop to 0 at the ends of the lens.

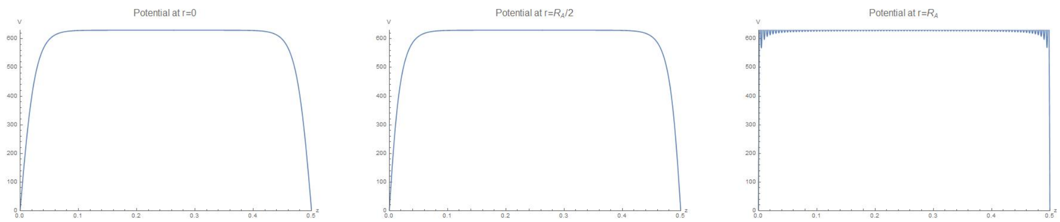

Figure 7: Plot of the potential against the longitudinal z as given by the analytical solution for a constant radius at $r = 0$ (left), $r = \frac{R_A}{2}$(middle) and $r = R_A$(right). The plots sum up to 50 odd n terms.

4.3 Longitudinal Trajectory

All we need to do is to subtract the calculated Anode potential with the plasma potential to get the overall potential. We then take the negative derivative with respect to the z-direction to get the electric field in the z-direction. Substituting this result to the equation of motion in the longitudinal direction would give the motion of the electron.

Giving a sketch of this mathematically, our overall potential is:

$$V(r, z) = V_A(r, z) - V_p(r)$$

We can then obtain the electric field longitudinally as:

$$E_z(r, z) = -\frac{\partial V(r, z)}{\partial z}$$

We plug this into the equation of motion from Eq. 2:

$$
\begin{aligned}
m_e \ddot{z} =& -e \times \left(-\frac{\partial V(r,z)}{\partial z} + (\dot{x}B_y - \dot{y}B_x) \right) \\
=& e \times \frac{\partial V(r,z)}{\partial z}
\end{aligned}
$$

where we will continue with the assumption of having a perfectly longitudinal magnetic field. It is worth noting here that the electric field only depends on how the anode potential varies along the longitudinal direction as we assumed a cold homogeneous plasma. The plasma potential is essentially a constant along the z-direction. But it is important to consider the sign of the potential, as when we fill the lens with more electrons than our found confinement condition, then the electric field's sign should be switched to reflect the fact that electrons should be expelled rather than be confined. Fig. 8 shows the longitudinal electric field resulting from the anode.

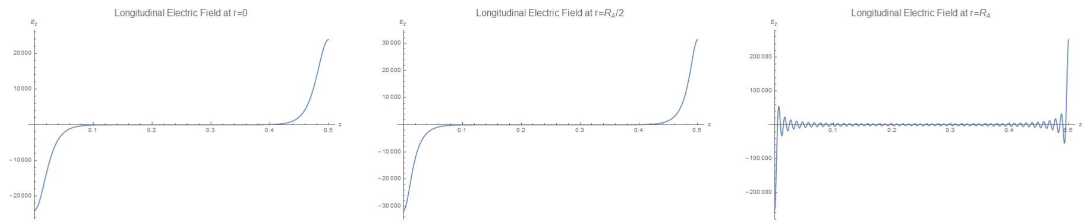

Figure 8: Plots of the longitudinal electric field when taking the derivative of the anode potential. The picture of the left shows the electric field at a constant $r = 0$, the middle at $r = \frac{R_A}{2}$ and the right at $r = R_A$.

To get an idea of the electron motion longitudinally we can plug in some values. We will start with an electron with an initial position $(0, 0, \frac{\ell}{2})$, with velocity $(0, 0, 10^{-4}$ MeV), with the same values given as in the radial confinement case at $0.9 \times$ Brillouin Flow Limit. This is shown in

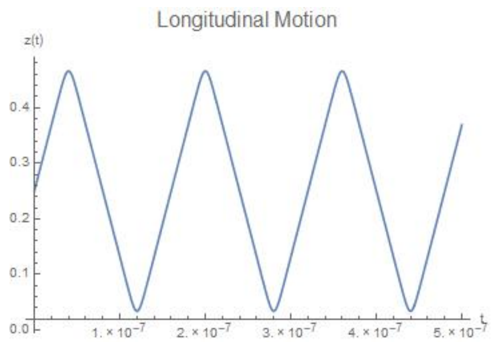

Figure 9: Longitudinal motion at $0.9 \times$ Brillouin Flow Limit at an initial position of $(0, 0, \frac{\ell}{2})$ and initial velocity $(0, 0, 10^{-4}$ MeV). Where we set $\ell = 0.5$ m, $R_A = 35 \times 10^{-3}$ m, $V = 630$ V, $B_z = 0.008$ T.

4.4 Plasma Potential Modification due to Endcaps

There may be some loss between the anode and the ground electrode which will have the effect of lowering the maximum radial expansion of the plasma (R_p) to the radius of the ground electrode. If we wanted to include such an effect we would have to use another definition for our plasma potential.

At $r < R_p$ the potential proceeds as in the earlier consideration:

$$V_{r,\mathrm{in}}(r) = -\int E(r)dr = \frac{en_e r^2}{4\epsilon_0} + C_1$$

where C_1 represents an integration constant. While for $r > R_p$:

$$V_{r,\mathrm{out}}(r) = -\int \left(-\frac{en_e {R_p}^2}{2\epsilon_0 r}\right) dr = \frac{en_e {R_p}^2}{2\epsilon_0} \ln(r) + C_2$$

where C_2 is an integration constant. At $r = R_A$ the potential should be equal to that of the anode:

$$V_A = \frac{en_e {R_p}^2}{2\epsilon_0} \ln(R_A) + C_2 \qquad \Rightarrow \qquad C_2 = V_A - \frac{en_e {R_p}^2}{2\epsilon_0} \ln(R_A)$$

giving the overall expression:

$$V_{r,\mathrm{out}}(r) = V_A - \frac{en_e {R_p}^2}{2\epsilon_0} \ln\left(\frac{R_A}{r}\right)$$

Next at $r = R_p$ we expect the potential to match (i.e. $V_{r,\mathrm{in}}(R_p) = V_{r,\mathrm{out}}(R_p)$):

$$\frac{en_e {R_p}^2}{4\epsilon_0} + C_1 = V_A - \frac{en_e {R_p}^2}{2\epsilon_0} \ln\left(\frac{R_A}{R_p}\right)$$

$$\Rightarrow C_1 = V_A - \frac{en_e {R_p}^2}{2\epsilon_0} \ln\left(\frac{R_A}{R_p}\right) - \frac{en_e {R_p}^2}{4\epsilon_0}$$

Putting this together, we get for the potential of the plasma:

$$\boxed{V_{r,\mathrm{in}}(r) = V_A - \frac{en_e}{4\epsilon_0}\left({R_p}^2 - r^2\right) - \frac{en_e {R_p}^2}{2\epsilon_0} \ln\left(\frac{R_A}{R_p}\right)} \tag{24}$$

To get the condition for maximal longitudinal confinement we use the fact that $V_{r,\mathrm{in}}(0) = 0$:

$$\begin{aligned} 0 =& V_A - \frac{en_e}{4\epsilon_0}{R_p}^2 - \frac{en_e {R_p}^2}{2\epsilon_0} \ln\left(\frac{R_A}{R_p}\right) \\ =& V_A - \frac{en_e {R_p}^2}{4\epsilon_0}\left(1 + 2\ln\left(\frac{R_A}{R_p}\right)\right) \end{aligned}$$

$$\boxed{n_e = \frac{4\epsilon_0 V_A}{e{R_p}^2\left(1 + 2\ln\left(\frac{R_A}{R_p}\right)\right)}} \tag{25}$$

As a sanity check, if we set $R_p = R_A$, we get back to what we previously found. The code won't assume this modification.

5 Gabor Lens Focusing

We assume that the electron plasma provides a radial focusing force, i.e. $F_r = qE_r$. We are interested in how the proton beam would be focused by this electron cloud, so the trajectory of an ion is given by:

$$m_i \ddot{r} = qE_r \tag{26}$$

where m_i is the mass of the ion and q its charge, r the radial position of the ion, and E_r the radial electric field which again is defined from Eq. 4. Since our interest is in the focusing per unit length instead of time we need to do a change of variables. For an ion moving with a constant longitudinal velocity v_i, the path length is given by $z = v_i t$.

$$\begin{aligned} r' =& \frac{dr}{dz} = \frac{dr}{dt}\frac{dt}{dz} = \dot{r}\frac{dt}{dz} = \dot{r}\left(\frac{1}{v_i}\right) \\ r'' =& \frac{dr'}{dz} = \frac{dr'}{dt}\frac{dt}{dz} = \ddot{r}\left(\frac{1}{{v_i}^2}\right) + \frac{\dot{r}}{v_i}\underbrace{\left[\frac{d}{dt}\left(\frac{1}{v_i}\right)\right]}_{\Rightarrow 0} \end{aligned}$$

So this change of variables makes the equation of motion become:

$$m_i\left(r''{v_i}^2\right) = -\,q\frac{n_e e}{2\epsilon_0}r$$

$$r'' + \underbrace{q\frac{n_e e}{2{v_i}^2 m_i \epsilon_0}}_{-{k_G}^2} r = 0$$

Under the thin lens approximation ($k_G\,\ell < \frac{\pi}{2}$) we can obtain the focal length by: $f^{-1} = {k_G}^2\ell$, where ℓ is the length of the lens, such that:

$$\begin{aligned} \frac{1}{f_G} &= -\frac{qen_e\ell}{2m_i{v_i}^2\epsilon_0} = \frac{en_e\ell}{4\epsilon_0 U} \\ &\boxed{f_G = \frac{4\epsilon_0 U}{en_e\ell}} \end{aligned} \tag{27}$$

where we define the total accelerating potential of the ion as $U = -\frac{1}{2}\frac{m_i{v_i}^2}{q}$. If we are not working in the thin lens limit then the focal length is instead given by:

$$\boxed{f_G = \frac{1}{k_G \sin(k_G\ell)}} \tag{28}$$

5.1 Visualization

Following the arguments made to obtain the focal length of the Gabor lens, it is best to see this visually to check that what we have done makes sense. We choose a magnetic field $B_z = 0.008$ T, which leads to a electron number density at the Brillouin flow limit of $n_e \sim 3.10867\times10^{14}\ \frac{\text{electrons}}{m^3}$. We will set the Gabor lens to be of length $\ell = 500\times10^{-3}$ m, and for the proton to be moving in the longitudinal direction with a constant velocity corresponding to 1 MeV. We will also

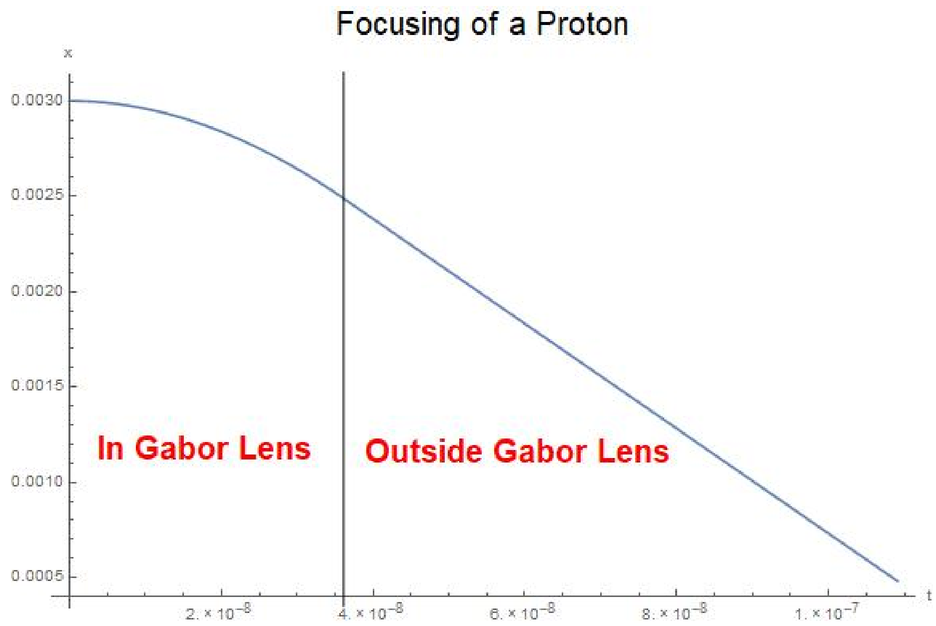

Figure 10: A plot of a proton initially at $(3 \times 10^{-3}\ \text{m}, 0, 0)$ at $t = 0$ at the entrance to the Gabor lens with a magnetic field $B_z = 0.008$ T, length $\ell = 0.5$ m, and with an electron density at the Brillouin flow limit. The resulting trajectory in the x-direction of the proton is plotted against time, with the vertical line indicating when the proton leaves the Gabor lens with the subsequent change in the trajectory (occurs at $t \sim 3.61 \times 10^{-8}$ s). The plot ends at the time predicted to correspond to when the proton reaches focal length of the lens at $t \sim 1.09 \times 10^{-7}$ s.

initially place the proton at the entrance of the lens at position: $(3 \times 10^{-3}\ \text{m}, 0, 0)$, such that we have slightly displaced the proton in the x-direction from the optimal center in order to observe focusing.

If we substitute the values into Eq. 28, it will give the focal length to be at about 1.51 m in the longitudinal direction. To check this, we let *Mathematica* numerically solve the differential equations and work out the velocity and position of the proton when it exits the Gabor lens at $z = 500 \times 10^{-3}$ m. The proton will then continue moving longitudinally. However, in the x-direction, it will now move with a constant velocity. The resulting trajectory for the x-position of the proton is given in Fig. 10. We have plotted the x-position against time. The plot stops at the time which corresponds to the predicted focal length (i.e. $t \sim 1.09 \times 10^{-7}$ s). Based on the resulting plot, it gives a reasonable result which appears to agree with what the theory suggests, allowing for some deviations due to the numerical integration performed.

6 Additional Clarification

6.1 General Method and Cylindrical Symmetry

To explain the theory I assumed a cylindrical symmetry quite early on. This is a bit contrary to the purpose of the code which is to try to get the motion without such an early assumption. The trouble in working purely in Cartesian coordinates is due to the electric field. As a reminder we assumed throughout the plasma is cold and homogeneous such that it would only act radially and by using Gauss's law we implicitly assume an infinitely long cylinder. To avoid doing so, we can instead work by using the solution to Poisson's equation to get the potential of the plasma.

$$\nabla^2 V_p = -\frac{\rho}{\epsilon_0} \tag{29}$$

Writing this out in Cartesian coordinates:

$$\frac{\partial^2 V_p}{\partial x^2} + \frac{\partial^2 V_p}{\partial y^2} + \frac{\partial^2 V_p}{\partial z^2} = \frac{en_e}{\epsilon_0}$$

while in Cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V_p}{\partial \phi^2} + \frac{\partial^2 V_p}{\partial z^2} = \frac{en_e}{\epsilon_0}$$

The problem then essentially comes down to applying the correct boundary conditions appropriate to the situation to get the appropriate plasma potential. It is reassuring that if we plug in what we previously found for V_p, it will satisfy Poisson's equation. So everything mentioned above still holds.

If we were to work with absolutely no consideration for cylindrical symmetry, it seems we would have to solve Laplace's equation for the anode potential, and solve Poisson's equation for the plasma potential. From those potentials, we would obtain the electric fields which we can substitute into Eq. 2.

6.2 Anode Potential Effect Radially

It may be noticed that when we considered the radial effects we did not include any consideration for the anode. It is worth seeing what effect the anode potential will have on the radial direction. Fig. 11 gives the electric field along the x-direction from the anode potential alone and the electron cloud plasma alone. The plots were plotted using the same parameters that have been used throughout. As can be seen, the effects of the anode can essentially be ignored assuming a high enough electron density. The plots of the anode contain some structure which comes from the numerical method to obtain the anode potential which may be ignored.

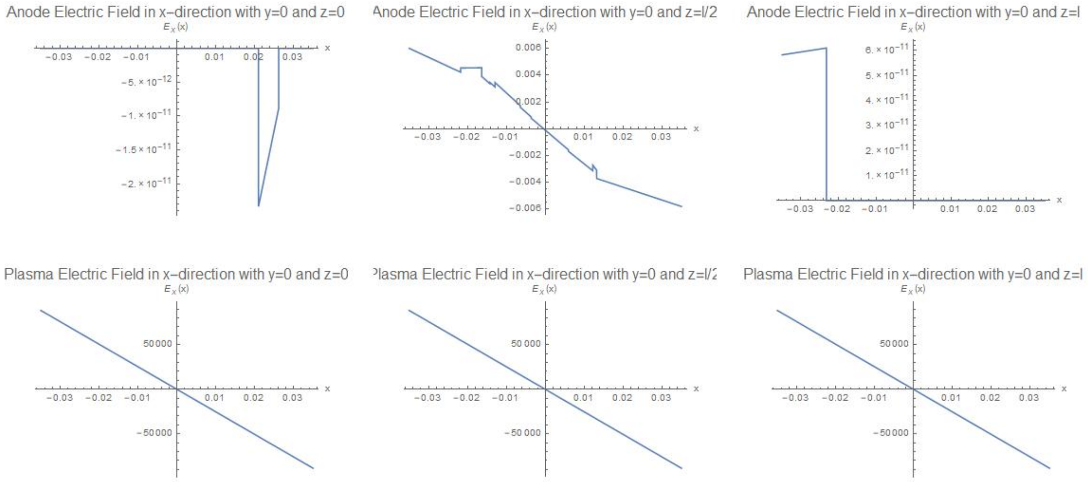

Figure 11: Plot of the electric field along the x-direction plotted against the x-coordinate. Due to symmetry we would get similar plots for the y-direction. The three plots above are the electric fields resulting from the anode alone, and the bottom plots resulting from the electron cloud alone. This is where the electron density is at $0.9 \times$ Brillouin Flow Limit, $\ell = 0.5$ m, $R_A = 35 \times 10^{-3}$ m, $V = 630$ V, $B_z = 0.008$ T. The structure in the top plots can be attributed to the numerical method used to obtain the anode potential.

7 *Mathematica* Code

7.1 Implementation

The current state of the code is to be able to reproduce the results of the theory which assumes cylindrical symmetry, but does so through Cartesian coordinates. In theory, my changing the boundary conditions for the potentials, it should be possible to get some general results without such symmetry.

The basic implementation of the code can be summarised as:

1. State the lens parameters, applied magnetic field, and the electron number density.
2. Solve Laplace's equation numerically to get the anode potential with appropriate boundary conditions.
3. Solve Poisson's equation with the chosen electron density to get the potential from the plasma with appropriate boundary conditions.
4. Take the derivative of the potentials to get the electric fields in the x, y, z-directions.
5. Substitute these into the equations of motion and solve for motion in the x, y, and z-direction.

7.2 Running the code

The code has been labeled with sections and some functions have comments to explain the functions. Some detail and how to run the code will be explained. But the whole notebook can be run as is and some default results will be given regardless.

7.2.1 Convenience Functions

These functions are meant for convenience such that the details beyond the implementation can be ignored to get results. Just run the cell with (Shift+Enter). Some of the more important bits codes will be briefly explained.

```
v[ke_, rm_: 0.511]
```

- Converts MeV into $\frac{\mathrm{m}}{\mathrm{s}}$.
- First argument is energy of particle in MeV
- Optional second argument defaults to the electron rest mass in MeV. Can be specified with another rest mass for another particle. (i.e. 938.27 for a proton)

```
myNDSolve[{x0_, y0_, z0_}, {vx0_, vy0_, vz0_}, mi_: m, ei_: e, prec_: 10^8]
```

- This functions numerically solves for the position of particle. It only gives the result which will likely be of little use if used alone. See the next function which makes use of this function as well as providing plots.
- First argument specifies the initial position of particle in x,y, and z.
- Second argument specifies the initial velocity of the particle
- Optional third argument specifies mass of particle in kg. (defaults to electron mass)

- Optional fourth argument specifies charge of particle. The convention of the code is that e is the electron charge, so a proton's charge in this code will be $-e$. (defaults to electron charge)
- Optional fifth argument specifies the maximum number of steps for *Mathematica* to take. If code encounters stiffness issues, increasing this may help. (defaults to 10^8)

```
plotNDSolve[{x0_, y0_, z0_}, {vx0_, vy0_, vz0_}, type_: "", mi_: m,
    ei_: e, prec_: 10^8, step_: 10^-10]
```

- Does everything the previous function does but will also plot the results to visualize the trajectory.
- First argument specifies the initial position of particle in x,y, and z.
- Second argument specifies the initial velocity of the particle
- Optional third argument specifies the type of plot to show. 3 options that can be used.
  1. "3d": 3D plot of the trajectory.
  2. "Hist": Histogram of trajectory positions.
  3. Anything else will give a 2D parametric plot of the radial trajectories, a plot of the longitudinal trajectory against time, and a plot of the radial trajectories against time.

  (defaults to the 2D plots)
- Optional fourth argument specifies mass of particle in kg. (defaults to electron mass)
- Optional fifth argument specifies charge of particle. The convention of the code is that e is the electron charge, so a proton's charge in this code will be $-e$. (defaults to electron charge)
- Optional sixth argument specifies the maximum number of steps for *Mathematica* to take. If code encounters stiffness issues, increasing this may help. (defaults to 10^8)
- Optional seventh argument specifies the steps the increment of steps taken in obtaining points for plotting. Decreasing will give a more detailed graph but will slow performance. (defaults to 10^{-10})

7.2.2 Electron Density Cells

If you are only interested in running the code and changing the time, position, and velocity just run all the cells until you get to `Static Plots`. Specify the time with `tfinal` and modify `plotNDSolve`.

Some additional details are provided below if you want to modify the default code.

Lens Parameters: Specifies some of the lens parameters.

Magnetic Field: Specifies the magnetic field.

Constants: Various constants as well as the electron density represented by η

Longitudinal Potential: Cell block to obtain the potentials.

- The format of the cell is first I state the Laplacian and which dimensions. Next will be the boundary conditions relevant to the situation. The last part will attempt to numerically solve for the potential for the given dimensions specified above.

- For the plasma potential I've included a commented out section which will solve in 3 dimensions, allowing for the potential to vary longitudinally, which can be run if interested.
  - To do so, remove the (∗ and ∗) around the code
- After the two potentials have been found, combine the two and take the derivatives along the three directions respectively to get the electric field.
  - In the radial direction I've had it depend solely on the plasma potential and ignored the anode for the reasons stated in the previous section and due to numerical issues near the border due to the anode's solution.
  - In the longitudinal direction I've defined it as a piecewise to reflect a situation when the electron density is high enough to dominate over the anode.

Static Plots: Plotting results.

- Here `tfinal` specifies until what time to solve for. Change this variable as desired.
- Change the arguments to `plotNDSolve` as stated previously.

Dynamic Plots: Allows for dynamically altering values if you don't want to mess with the code but is not very good performance wise and does not include any considerations for boundaries.

7.2.3 Proton Focusing

I've included some code to display the proton focusing. This runs much like the electron code above, but the trajectories are for the proton not an electron. Not the main purpose of the code, so it is more for display purposes. Included a comparison to the estimated focal length from the theory and show's the focusing of the lens.

7.3 Results

A quick summary of some of the results the code can produce. Assume default parameters lens length $\ell = 500 \times 10^{-3}$ m, anode radius $R_A = 35 \times 10^{-3}$ m, anode voltage $V = 630$ V, magnetic field $B_z = 0.008$ T, electron density at $0.9 \times$ Brillouin Flow Limit, initial position $(0, 0, \frac{\ell}{2})$ and initial velocity $(1\text{ eV}, 1\text{ eV}, 100\text{ eV})$.

2D Plots

Can output 2D plots, the left is for the radial motion, the middle one is the longitudinal motion against time, and the one on the right is of the radial trajectories plotted against time.

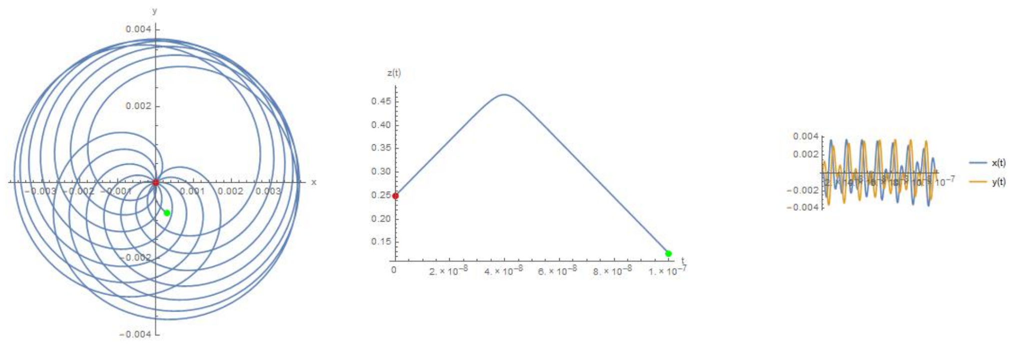

Figure 12: Solves until a time $t = 10^{-7}$ s

The code can vary the time to which it will solve for, increased time to $t = 10^{-6}$ s.

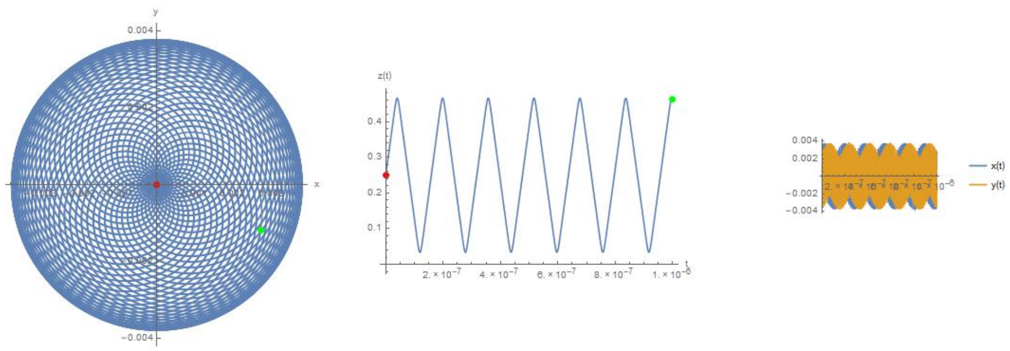

Figure 13: Increased time to $t = 10^{-6}$ s.

3D Plots

Code can output a 3D plot:

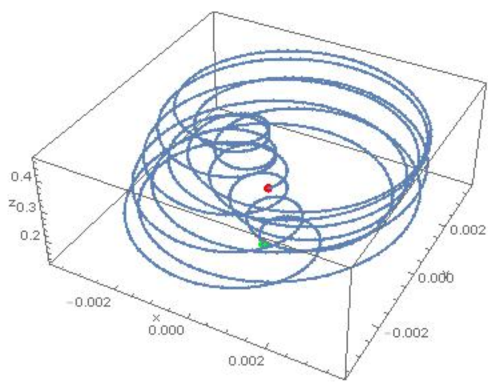

Figure 14: Solves until a time $t = 10^{-7}$ s.

Histogram

Can also output a histogram:

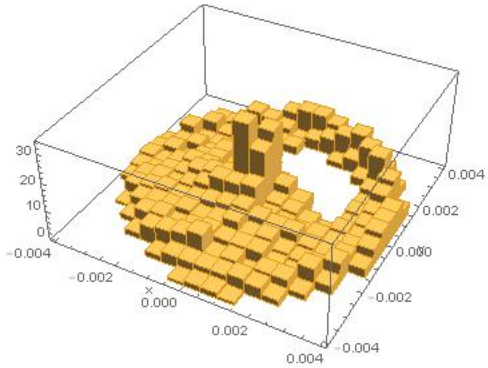

Figure 15: Solves until a time $t = 10^{-7}$ s.

Proton Focusing

Can plot 2D, 3D, and histograms much like above. Also included focal length as compared with the theory:

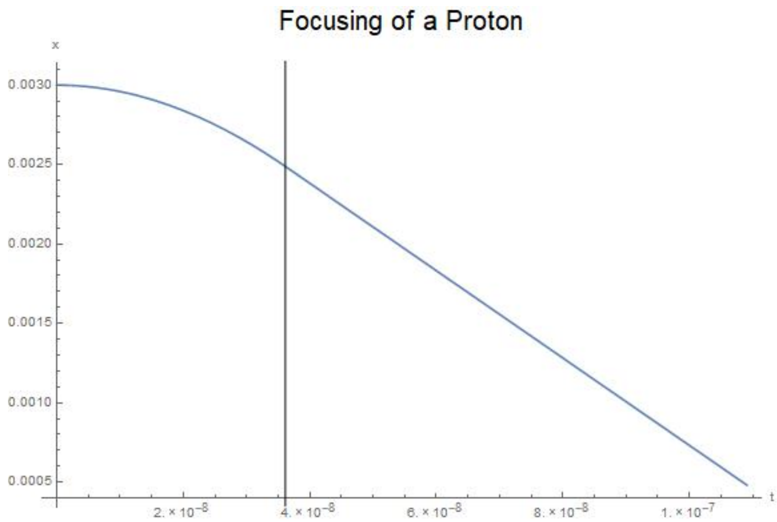

Figure 16: When electron density at the Brillouin flow limit, proton escapes at $\sim 3.6 \times 10^{-8}$ s.

Boundaries

Code includes considerations for boundaries to stop integrations once a boundary is reached.

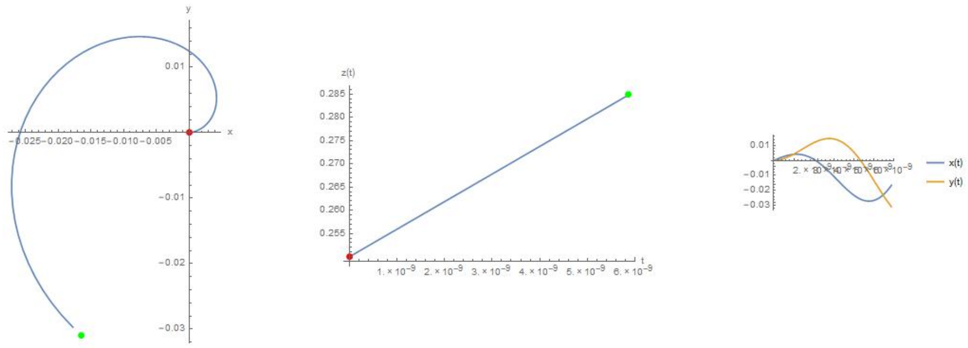

Figure 17: Increased velocity to an initial value of (100 eV, 1 eV, 100 eV). Stops when it reaches a radius matching the anode radius at a time $\sim 5.869 \times 10^{-9}$ s. (output in the code but not shown in the image)

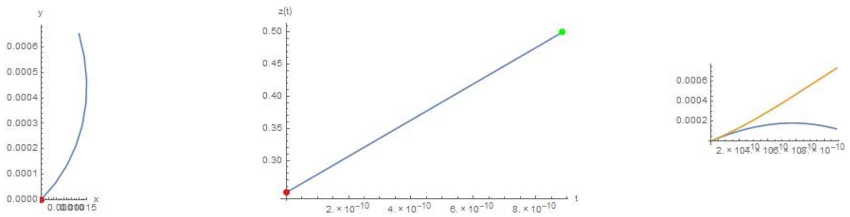

Figure 18: Increased velocity to an initial value of (1 eV, 1 eV, 1 MeV). Stops when it escapes in the forward direction at $\sim 8.85 \times 10^{-10}$ s.

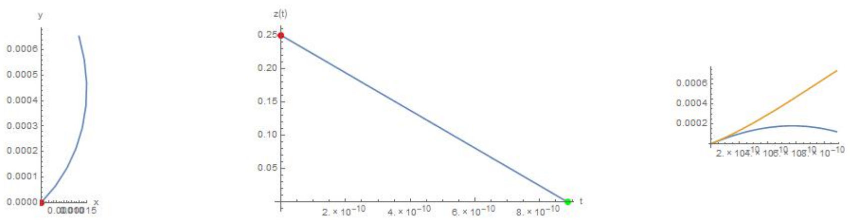

Figure 19: Increased velocity to an initial value of (1 eV, 1 eV, −1 MeV). Stops when it escapes in the backward direction at $\sim 8.85 \times 10^{-10}$ s.

Changing Electron Density

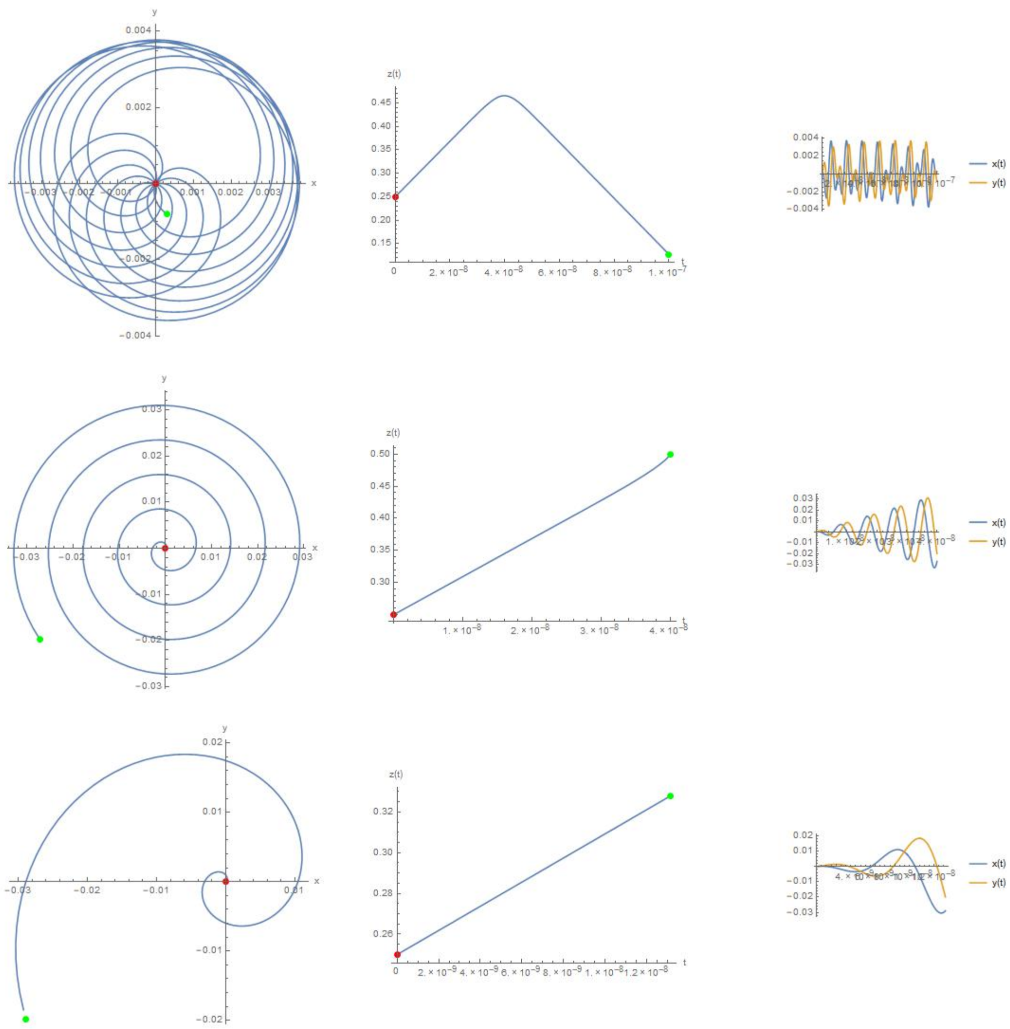

Figure 20: Results when at 0.9 × Brillouin Limit(top), 1.0 × Brillouin Limit(middle), 1.1 × Brillouin Limit(bottom). At higher electron density, the code will stop at earlier times, so the plots are not at equal time.

Changing Magnetic Field

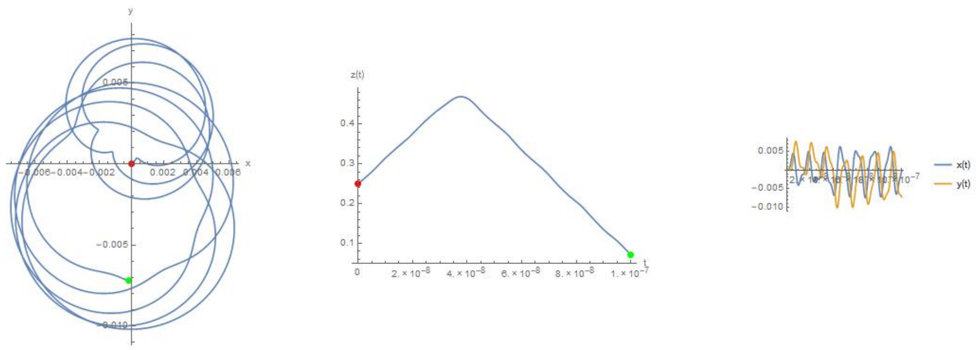

Figure 21: Changing magnetic field to include an x-component: $(0.001\ \mathrm{T}, 0, 0.008\ \mathrm{T})$, solving to a time $t = 10^{-7}$ s.

Dynamical Plot

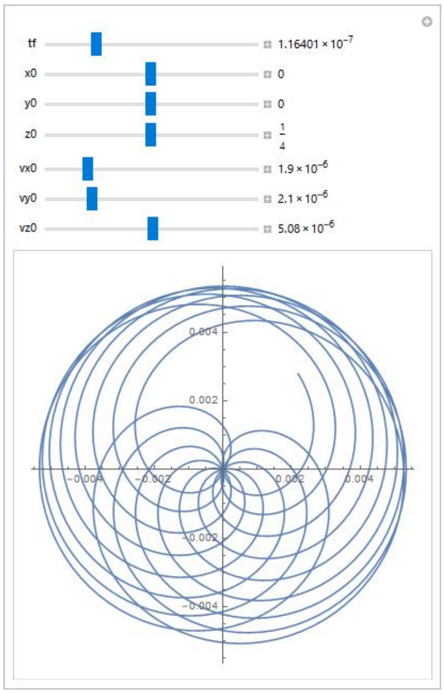

Figure 22: Can dynamically edit values if desired, though performance wise does not run too well.