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R. H. Levy

Citation: *The Physics of Fluids* **8**, 1288 (1965); doi: 10.1063/1.1761400

View online: <https://doi.org/10.1063/1.1761400>

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## Diocotron Instability in a Cylindrical Geometry

R. H. LEVY

*Avco-Everett Research Laboratory, Everett, Massachusetts*  
(Received 11 December 1964)

The diocotron (or slipping stream) instability of low density ( $\omega_p \ll \omega_c$ ) electron beams in crossed fields is considered for a cylindrical geometry. For a simple density distribution, the normal modes of the electron beam correspond to a continuum of eigenvalues, plus two discrete eigenvalues. Work due to Case and Dikii appears to show that the continuous spectrum is not important in stability studies of this type. The condition for stability considering the discrete modes only is derived; under suitable geometrical and electrical conditions, it is shown that these modes can be stable. The analogy between the electromagnetic problem considered here and the problem of the stability of an ideal rotating fluid is discussed. It is shown that stability conditions derived for the latter problem depend on the possibility of axial perturbations; what this implies for the electron beam problem is briefly discussed.

### INTRODUCTION

THE Diocotron (or slipping stream) instability has been known for some time,<sup>1-4</sup> and it forms the basis of the small-signal theory of the crossed field microwave magnetron. For one reason or another, however, it appears that this instability has not been extensively studied in a cylindrical geometry. This situation, although somewhat surprising at first sight (since magnetrons are generally cylindrical) may possibly be explained by the observation that the annulus in which the electron beam travels frequently has a rather small aspect ratio and can therefore be approximately treated as planar. Whatever the situation in this regard, certain phenomena relevant to thick beams in cylindrical geometries are not adequately treated by the planar theory. Thus, it is known from the planar theory of thick electron beams that such beams are always unstable to perturbations having sufficiently long wavelengths. However, it has been pointed out<sup>5</sup> that when a thick beam is moving around a circular (or other closed) path, that an upper limit to the wavelength of permissible disturbances is approximately given by the perimeter of the path. Thus, the question arises as to whether an electron beam moving in a circular path in crossed electric and magnetic fields can be stabilized by being made sufficiently thick. It is the purpose of this note to give a quantitative evaluation of this effect.

<sup>1</sup> G. C. MacFarlane and H. G. Hay, Proc. Roy. Soc. (London) **63B**, 409 (1953).

<sup>2</sup> O. Buneman, in *Crossed Field Microwave Devices*, edited by E. Okress (Academic Press Inc., New York 1961), Chap. 5.

<sup>3</sup> R. W. Gould, Electron Tube and Microwave Laboratory, California Institute of Technology, Technical Report No. 3 (1955).

<sup>4</sup> B. Epsztein, Compt. Rend. **240**, 408 (1955).

<sup>5</sup> W. Knauer, Hughes Research Laboratory, Research Report 332 (1960).

### BASIC FORMULATION

We consider the geometry illustrated in Fig. 1. Two concentric, perfectly conducting cylinders of radii  $a$  and  $d$  are aligned along the  $z$ -axis. A constant uniform magnetic field of strength  $B$  acts in the  $z$ -direction. In the basic (unperturbed) state the space between the electrodes is filled with electrons having a density  $n_0(r)$  where  $r$  is the distance from the axis. Following Gould's analysis of the planar case, we suppose that the electron density is sufficiently low relative to the magnetic field intensity that  $\omega_p \ll \omega_c$ , the symbols referring, respectively, to the plasma and cyclotron frequencies. The unperturbed state is then defined by a radial electric field  $E_0(r)$  which is related to the electron density by Gauss' law:

$$\frac{1}{r} \frac{d}{dr} (rE_0) = \frac{-n_0 e}{\epsilon_0}. \quad (1)$$

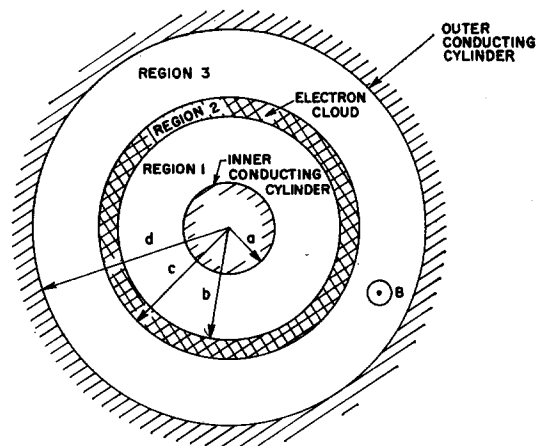


FIG. 1. An illustration of the basic geometry considered in the text.

The electric charge on the inner electrode (per unit axial length) is just

$$Q = 2\pi a \epsilon_0 E_0(a). \quad (2)$$

$Q$  can be equal in magnitude (but opposite in sign) to the total charge in the electron cloud, but it can also have any other value. Each value of  $Q$  corresponds to a definite value of the potential between the inner and outer cylinders. The electrons move (in the unperturbed state) in the azimuthal direction with velocity  $v_0 = -E_0/B$ .

We consider next the perturbed motions of this system. To start with, we consider only two-dimensional perturbations, but we give a brief discussion of three-dimensional perturbations at a later stage. Once again following Gould, we apply the quasi-static approximation and assume that the electric field due to any perturbation can be treated as irrotational. We anticipate the result that the frequencies of interest in this study are on the order of  $\omega_p^2/\omega_c$ , that is, much less than  $\omega_p$  and hence, *a fortiori*, much less than  $\omega_c$ . This observation justifies taking for the electronic equation of motion

$$E_r = -vB, \quad E_\theta = uB, \quad (3)$$

where  $(u, v)$  and  $E_r, E_\theta$  are, respectively, the radial and azimuthal components of the velocity and electric fields. The quasi-static assumption implies first the existence of a potential  $\phi$ ,

$$E_r = -\frac{\partial\phi}{\partial r}, \quad E_\theta = -\frac{1}{r} \frac{\partial\phi}{\partial\theta}, \quad (4)$$

and second, from Eq. (3),

$$\frac{1}{r} \frac{\partial}{\partial r} (ru) + \frac{1}{r} \frac{\partial v}{\partial\theta} = \text{div } \mathbf{v} = 0. \quad (5)$$

In addition to the above, we have the equation of conservation for the electrons:

$$\partial n/\partial t + n \text{ div } \mathbf{v} + \mathbf{v} \cdot \nabla n = 0. \quad (6)$$

In view of Eq. (5), the middle term in Eq. (6) vanishes; thus

$$\frac{Dn}{Dt} = \frac{\partial n}{\partial t} + \mathbf{v} \cdot \nabla n = \frac{\partial n}{\partial t} + \frac{1}{Br} \frac{\partial(\phi, n)}{\partial(r, \theta)} = 0. \quad (7)$$

This condition states simply that the electron density of any small parcel is conserved following the motion, even though the density varies spatially or temporally.

We now linearize by assuming

$$\begin{aligned} \phi &= \phi_0(r) + \phi(r) \exp \{i(l\theta - \omega t)\}, \\ n &= n_0(r) + n(r) \exp \{i(l\theta - \omega t)\}, \end{aligned} \quad (8)$$

where, as usual, the physical quantities are the real parts of the complex quantities appearing in Eq. (8). On linearization, Eq. (7) yields

$$n \left( \omega - \frac{lv_0}{r} \right) = -\frac{l\phi}{Br} \frac{dn_0}{dr}. \quad (9)$$

Substituting Eq. (9) in Poisson's equation yields, finally,

$$\left( \omega - \frac{lv_0}{r} \right) \left\{ \frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) - \frac{l^2}{r^2} \phi \right\} = \frac{-e}{\epsilon_0} \frac{l\phi}{Br} \frac{dn_0}{dr}. \quad (10)$$

Up to this point, we have left the choice of zero-order profile entirely free. We shall now make a choice governed by considerations of convenience. We assume

$$\begin{aligned} n_0 &= 0 \quad (a \leq r < b; c < r \leq d), \\ n_0 &= N \quad (b \leq r \leq c). \end{aligned} \quad (11)$$

The purpose of this choice is that it makes  $dn_0/dr = 0$  in each of three regions. In the interior of these regions, then, Eq. (10) reduces to the much simpler form

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) - \frac{l^2}{r^2} \phi = 0 \quad (12)$$

and we also have, from Eq. (9),  $n = 0$ . Thus, the perturbation we have to deal with is much simplified and involves (as noted by Gould) no perturbation charge density at all in the interior of the electron cloud, but merely an accumulation at each of the two free surfaces. This observation leads us to consider the conditions to be applied across the free surfaces  $r = b$  and  $r = c$ . In the first place, we must clearly assume the perturbation potential to be continuous across these surfaces. For obtaining the change in  $d\phi/dr$  across the surface, various methods have been proposed, but the one that seems simplest is as follows: we merely integrate Eq. (10) for a short distance from  $r = b - \delta$  to  $r = b + \delta$  and let  $\delta \rightarrow 0$ . The bracket containing  $\omega$  has virtually a constant value in this range and can therefore be taken out of the integration. On the right-hand side,  $dn_0/dr$  can be treated as a delta function, while  $\phi$  (and  $\int \phi dr$ ), being continuous, give no contribution to an integral over a vanishing range. Putting these facts together yields

$$\left( \omega - \frac{lv_0(b)}{b} \right) \left\{ \frac{d\phi}{dr} \Big|_{b+} - \frac{d\phi}{dr} \Big|_{b-} \right\} = \frac{-\omega_p^2}{\omega_c} \frac{l\phi(b)}{b}. \quad (13)$$

The specification of the problem is now completed by noting that the boundary conditions appropriate to conducting electrodes at  $r = a$  and  $r = d$  are simply  $\phi(a) = \phi(d) = 0$ .

At this stage, it would appear that the problem is completely solved, at least in principle. We have only to write down the eigenfunctions which satisfy Eq. (12) in the three regions, apply the boundary and jump conditions, and derive the characteristic equation. In the present case, the characteristic equation will have the form of a polynomial in  $\omega$ , the degree of the polynomial corresponding to the number of surfaces at which  $n_0(r)$  is discontinuous. This can also be explained by noting that a surface wave can propagate at each discontinuity, so that clearly the number of such waves is just the number of such surfaces. For the unperturbed density profile described in Eq. (11), this number is just two. Since the coefficients of the polynomial are all real, the roots will be either real, or will occur in complex conjugate pairs. In the latter case, obviously, one root corresponds to a growing (unstable) wave and the other to an evanescent (damped) wave. Therefore, stability can only be claimed when *all* the roots of the characteristic polynomial are real, in which case each surface wave can propagate at constant amplitude.

It is clear, however, that the method described above cannot, as it stands, be used to make any firm statement about stability. This is because such a statement can only be made when we have obtained a complete set of normal modes; in the present case, we have a very restricted set corresponding in number to the number of surface discontinuities present in the unperturbed state. That this set is not complete is easily seen by observing that no initial condition involving a perturbation in the charge density can be described by them. Now an analogous problem has been extensively treated by Case<sup>6,7</sup> and Dikii<sup>8</sup> in connection with the problem of aerodynamic shear flow. This problem is mathematically identical to the slipping electron stream problem provided, as assumed here,  $\omega_p \ll \omega_c$ . Case points out that when  $dn_0/dr = 0$ , the solution of Eq. (10) can be written

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d\phi}{dr} \right) - \frac{l^2}{r^2} \phi = A \delta \left( \omega - \frac{lw_0}{r} \right), \quad (14)$$

where  $A$  is an arbitrary constant. The eigenfunctions corresponding to Eq. (14) give rise to a continuous spectrum of real eigenvalues, the spectrum covering all angular frequencies present in the unperturbed state. Each eigenfunction corresponds to a delta

function perturbation of charge density at what might be called the corresponding resonant layer. Case shows, in a particular case, by using the method of the Laplace transform, that perturbations involving these eigenfunctions decay at long times like various algebraic powers of the time. Therefore, the stability will depend only upon the behaviour of the discrete normal modes, that is, those picked out by the previous discussion. This proof is given in more general form by Dikii, and it is upon the validity of this proof that our work, together with that of Gould, and a large amount of earlier work in the field of aerodynamic shear flows, depends. Among the aerodynamic work, we particularly note the work of Goldstein<sup>9</sup> who considers a profile having no less than five discontinuities, and hence is forced to consider the roots of a quintic polynomial. This work points out that, if one is willing to undertake a large amount of tedious work, an arbitrary continuous profile of (say) electron density, can be satisfactorily approximated by a small number of segments in each of which the electron density has a different constant value.

Before leaving this point, we should perhaps insert a caution along the following lines: According to Dikii, the results obtained by a stability analysis of the flow of an inviscid fluid do in fact agree with the results obtained when a small viscosity is allowed, and then made to tend to zero. This is an important point, since the eigenfunctions corresponding to the continuous spectrum have discontinuous derivatives; these jumps cannot represent physical fact in a real medium. To smooth out the jumps, it is necessary to introduce more physics, and in the fluid case, this physics is just the viscosity. Dikii's observation is therefore of importance when interpreting the ideal stability analysis. In our medium, the jumps in the eigenfunctions are also not physically acceptable, however, smoothing them out is obviously not to be accomplished by the simple addition of a diffusivity, but would require consideration of the electron dynamics by means of a velocity distribution function. We therefore make the assumption (which seems plausible but no more than that) that Dikii's result is independent of the details of the physical process whose neglect resulted in the discontinuous eigenfunctions.

#### ESTABLISHMENT OF THE STABILITY CONDITION

No further difficulty of a theoretical nature remains at this stage, and we can proceed directly to

<sup>9</sup> S. Goldstein, Proc. Roy. Soc. (London) **A132**, 524 (1931).

<sup>6</sup> K. M. Case, Phys. Fluids **3**, 143 (1960).

<sup>7</sup> K. M. Case, Phys. Fluids **3**, 149 (1960).

<sup>8</sup> L. A. Dikii, Dokl. Akad. Nauk SSSR **135**, 1068 (1960) [English transl.: Soviet Phys.—Doklady **5**, 1179 (1960)].

write down the eigenfunctions, the dispersion relation, and the condition that both the roots of the latter should be real. The first step is to note the zero order potential and electric field distribution that are implied by the distribution of charge given in Eq. (11). Taking the conductor at  $r = a$  to be at zero potential, we find in region 1 that

$$E_0 = \frac{Q}{2\pi\epsilon_0 r}, \quad \phi_0 = -\frac{Q}{2\pi\epsilon_0} \ln \frac{r}{a}. \tag{15}$$

In region 2, we find

$$E_0 = \frac{Q}{2\pi\epsilon_0 r} - \frac{Neb}{2\epsilon_0} \left( \frac{r}{b} - \frac{b}{r} \right), \tag{16}$$

$$\phi_0 = -\frac{Q}{2\pi\epsilon_0} \ln \frac{r}{a} + \frac{Neb^2}{4\epsilon_0} \left\{ \frac{r^2}{b^2} - 1 - 2 \ln \frac{r}{b} \right\}.$$

In region 3, we find

$$E_0 = [Q - Ne\pi(c^2 - b^2)]/2\pi\epsilon_0 r, \tag{17}$$

$$\phi_0 = -\frac{Q}{2\pi\epsilon_0} \ln \frac{r}{a} + \frac{Ne}{4\epsilon_0} \cdot \left\{ (c^2 - b^2) + 2c^2 \ln \frac{r}{c} - 2b^2 \ln \frac{r}{b} \right\}.$$

The potential of the outer conductor, at  $r = d$  is related to the charge on the inner conductor by

$$\phi_0(d) = -\frac{Q}{2\pi\epsilon_0} \ln \frac{d}{a} + \frac{Ne}{4\epsilon_0} \cdot \left\{ (c^2 - b^2) + 2c^2 \ln \frac{d}{c} - 2b^2 \ln \frac{d}{b} \right\}. \tag{18}$$

The solutions of Eq. (12) are the simple functions  $r^{\pm l}$ . We therefore take for the eigenfunction in region 2,<sup>10</sup>

$$\phi = \beta r^l + \gamma r^{-l} \quad (l \geq 1), \tag{19}$$

where  $\beta$  and  $\gamma$  are arbitrary constants. The eigenfunction appropriate to region 1 must vanish at  $r = a$ , and be continuous with Eq. (19) at  $r = b$ . Thus,

$$\phi = (\beta b^{2l} + \gamma)(r^{2l} - a^{2l})(b^{2l} - a^{2l})^{-1} r^{-l}. \tag{20}$$

The eigenfunction appropriate to region 3 must vanish at  $r = d$ , and be continuous with Eq. (19) at  $r = c$ . Thus,

$$\phi = (\beta c^{2l} + \gamma)(d^{2l} - r^{2l})(d^{2l} - c^{2l})^{-1} r^{-l}. \tag{21}$$

The condition Eq. (13) on the jump in  $d\phi/dr$  at

<sup>10</sup> The mode  $l = 0$  has no nontrivial solution. This can be seen as follows: Eq. (13) shows that for this mode  $d\phi/dr$  as well as  $\phi$  is continuous at  $r = b, c$ . The eigenfunction  $\phi = A + B \ln r$  is therefore valid in all three regions. If  $\phi(a) = \phi(d) = 0, A = B = 0$ .

$r = b$ , together with the similar one at  $r = c$  now yield

$$2 \left( \omega + \frac{Ql}{2\pi N e b^2} \right) (\beta a^{2l} + \gamma) = (\beta + \gamma b^{-2l})(b^{2l} - a^{2l}), \tag{22}$$

$$\left[ 2 \left( \omega + \frac{Ql}{2\pi N e c^2} \right) - l \left( 1 - \frac{b^2}{c^2} \right) \right] (\beta d^{2l} + \gamma) = -(\beta + \gamma c^{-2l})(d^{2l} - c^{2l}). \tag{23}$$

In these equations, and hence forward, the unit of frequency has been taken to be  $\omega_b^2/\omega_c$ , or  $Ne/\epsilon_0 B$ . The dispersion relation is now obtained by writing down the condition for consistency of these two linear homogeneous equations in  $\beta$  and  $\gamma$ .

$$\begin{aligned} & -4\omega^2(d^{2l} - a^{2l}) \\ & + 2\omega \left[ l(d^{2l} - a^{2l}) \left\{ \left( 1 - \frac{b^2}{c^2} \right) - \frac{Q}{\pi N e b^2} \left( 1 + \frac{b^2}{c^2} \right) \right\} \right. \\ & \left. + (b^{2l} c^{2l} - a^{2l} d^{2l})(c^{2l} - b^{2l}) b^{-2l} c^{-2l} \right] \\ & + \left[ \frac{l^2 Q}{\pi N e b^2} \left( 1 - \frac{b^2}{c^2} - \frac{Q}{\pi N e c^2} \right) (d^{2l} - a^{2l}) \right. \\ & \left. - \frac{lQ}{\pi N e b^2} (c^{2l} - a^{2l})(d^{2l} - c^{2l}) c^{-2l} \right. \\ & \left. - l \left( 1 - \frac{b^2}{c^2} - \frac{Q}{\pi N e c^2} \right) (d^{2l} - b^{2l})(b^{2l} - a^{2l}) b^{-2l} \right. \\ & \left. + (c^{2l} - b^{2l})(d^{2l} - c^{2l})(b^{2l} - a^{2l}) b^{-2l} c^{-2l} \right] = 0. \end{aligned} \tag{24}$$

The condition for reality of the roots of this quadratic in  $\omega$  which is also the condition for stability of the distribution described, is now easily extracted. After some reduction, the condition for stability can be written as

$$\begin{aligned} & \left[ -l \left( 1 + \frac{Q}{\pi N e b^2} \right) \left( 1 - \frac{b^2}{c^2} \right) (d^{2l} - a^{2l}) \right. \\ & \left. + 2(d^{2l} + a^{2l}) - (c^{2l} + b^{2l}) \left( \frac{a^{2l} d^{2l}}{b^{2l} c^{2l}} + 1 \right) \right]^2 \\ & - \frac{4}{b^{2l} c^{2l}} (d^{2l} - c^{2l})^2 (b^{2l} - a^{2l})^2 \geq 0. \end{aligned} \tag{25}$$

### DEDUCTIONS FROM THE STABILITY CONDITION

Several simple deductions are possible from the stability condition, Eq. (25). Firstly, it is important to note that the condition Eq. (25) can always be fulfilled for any geometry by having a sufficiently

large positive or negative value of  $Q$ . Alternatively, the condition for instability will only be satisfied by a definite limited range of values of  $Q$  (or of the unperturbed potential between the conductors).

Secondly, we note that if either  $d = c$ , or  $b = a$ , Eq. (25) has the form of a perfect square, guaranteeing the satisfaction of the stability condition. The physical meaning of this is simply that if either edge of the beam is in contact with a fixed conductor, the wave that would normally be associated with that edge can now no longer exist. Alternatively, the dispersion relation for this case, if derived *ab initio*, is now simply a linear equation in  $\omega$ , and therefore incapable of having complex roots. In such a case, the system is capable of only one real frequency of oscillation for any  $l$ .

A simple limiting case of some interest involves letting  $d \rightarrow \infty$  (removing the outer conducting cylinder) and setting  $Q = \pi Ne(c^2 - b^2)$  so that the positive charge on the inner cylinder equals the negative charge in the electron cloud. This implies, from Eq. (16), that  $E_0$  vanishes for  $r \geq c$ . The condition for stability in these circumstances becomes

$$\left[ -l \left( \frac{c^2}{b^2} - 1 \right) + 2 - \frac{a^{2l}}{b^{2l}} \left( 1 + \frac{b^{2l}}{c^{2l}} \right) \right]^2 - 4 \frac{b^{2l}}{c^{2l}} \left( 1 - \frac{a^{2l}}{b^{2l}} \right)^2 \geq 0. \quad (26)$$

For simplicity we restrict our attention to the mode  $l = 1$ :

$$(c^2 - b^2)^2 (2bc - a^2 - c^2)(-2bc - a^2 - c^2) \geq 0. \quad (27)$$

The factor  $(c^2 - b^2)^2$  is always  $\geq 0$  and may be dropped. The last factor is always negative. The stability condition is thus, finally,

$$a^2 + c^2 \geq 2bc. \quad (28)$$

Regions of stability for this case for the mode  $l = 1$  and a few higher modes are shown in Fig. 2. In this case, the cylinder at  $r = a$  is at a positive potential relative to "infinity."

Another limiting case of greater interest for laboratory purposes is reached by setting  $a = 0$ , that is, removing the inner conductor. In addition, we must set  $Q = 0$  for consistency. In these circumstances, the stability condition reduces to

$$\left[ -l \left( 1 - \frac{b^2}{c^2} \right) d^{2l} + 2d^{2l} - c^{2l} - b^{2l} \right]^2 - 4b^{2l}c^{-2l}(d^{2l} - c^{2l})^2 \geq 0. \quad (29)$$

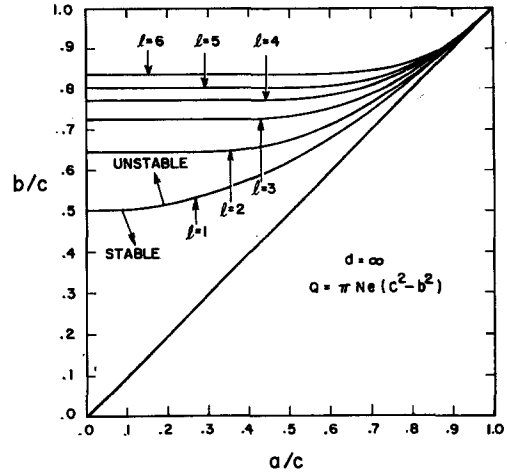


FIG. 2. For the case  $d \rightarrow \infty$ ,  $Q = \pi Ne(c^2 - b^2)$  this figure shows the geometric parameters governing the slipping stream instability. Since  $a \leq b \leq c$  only a triangle on this figure represents possible geometries. It can be seen that the  $l = 1$  mode is the most important. Note that the configuration is stable when  $a = 0$ ,  $c \geq 2b$ , and that it is unstable whenever  $b = c$ .

For  $l = 1$  this condition reduces to

$$(d^2 - c^2)^2 (c^2 - b^2)^2 \geq 0; \quad (30)$$

which condition is satisfied for all values of the parameters. This mode is therefore always stable. For  $l = 2$  the condition is

$$(c^2 - b^2)^2 [c^2(c^2 + b^2)^2 - 4b^2d^4] \geq 0, \quad (31)$$

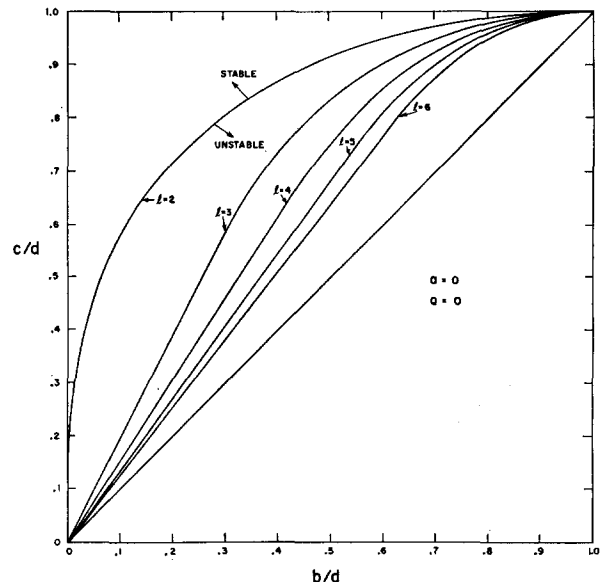


FIG. 3. For the case  $a = 0$ ,  $Q = 0$  this figure shows the geometric parameters governing the slipping stream instability. Since  $b \leq c \leq d$  only a triangle on this figure represents possible geometries. The  $l = 1$  mode cannot lead to instability in this geometry; the  $l = 2$  mode is therefore the most important. Note that when  $b = 0$  the configuration is stable for all values of  $c > 0$ , and that when  $b = c$  it is always unstable.

or, more simply,

$$c(c^2 + b^2) \geq 2bd^2. \tag{32}$$

Regions of stability for this case for the mode  $l = 2$  and a few higher modes are shown in Fig. 3.

The stability condition for the plane geometry illustrated in Fig. 4 can be derived from Eq. (25) by letting  $a, b, c, d$  all tend to infinity while keeping the differences between these lengths constant. However, in order to maintain a finite wavelength for the perturbation, it is necessary to let  $l$  tend to infinity as well, keeping  $k = l/a$  (the wavenumber of the perturbation in the stream direction) finite. In this way, one obtains a relation involving exponentials. It must also be observed that in this limit, one must replace  $Q$  by  $2\pi\sigma a$ , where  $\sigma$  is the charge density on the inner conductor. Then, in the limit, the term  $Q/\pi N e b^2 = 2\sigma a/N e b^2 \rightarrow 0$ , and the actual value of  $\sigma$  becomes irrelevant. This makes sense, since the electric fields in regions 1 and 3 for the plane case are constants. Without affecting the stability question, either field can be removed by transferring to a set of coordinates moving with appropriate velocity parallel to the beam. The difference between the two electric fields is important, however, and represents the velocity change across the beam. This velocity change can be shown to

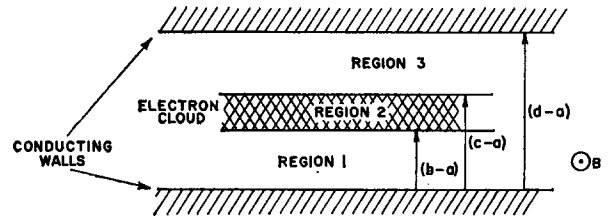


FIG. 4. An illustration of the manner in which the planar problem can be approached as a limiting case of the cylindrical problem.

be equal to  $\omega_p^2/\omega_0$  multiplied by the beam thickness. An important observation is that the plane case cannot be stabilized merely by applying a large positive or negative potential between the plates. We shall not consider this case further, as it has been rather thoroughly treated in the microwave and aerodynamic literature.

It is known from the plane case that thin beams are most unstable, and this leads us to consider the case  $(1 - b/c) \ll 1$ . If we set  $b = c$ , it is easily seen that the expression on the left-hand side of Eq. (25) vanishes identically, showing that the case  $b = c$  is marginally stable. More detailed study of this case is then necessary. One finds that for  $b/c$  slightly less than unity, the beam is or is not stable according as  $Q \leq -N e \pi b^2$ . The marginal condition corresponds to solid body rotation.

TABLE I. This table lists for various values of the ratios  $a/d, b/d$ , and  $c/d$ , and for various mode numbers, the range of charges on the inner cylinder, or of potentials across the two conducting cylinders, between which instability exists. For fixed geometry, the upper and lower potentials tend, with increasing mode number, to the same limit; this limit (which corresponds to solid body rotation of the electron beam) is shown as  $l = \infty$ . The unit of charge per unit length is  $N e \pi (c^2 - b^2)$ , the unit of potential is  $\frac{1}{2} N e e_0^{-1} (c^2 - b^2)$ .

$a/d$	$b/d$	$c/d$	$l$	$Q_1$	$Q_2$	$\phi_1$	$\phi_2$
0.1	0.2	0.3	1	2.75	.10	-4.95	1.15
			2	1.17	-.02	-1.33	1.42
			3	.43	-.13	.39	1.68
			4	.06	-.22	1.24	1.89
			$\infty$		-.8	3.22	
0.2	0.4	0.6	1	1.96	.04	-2.46	.62
			2	1.00	-.04	-.93	.75
			3	.39	-.14	.05	.91
			4	.05	-.23	.60	1.05
			$\infty$		-.8	1.97	
0.2	0.4	0.8	1	.24	-.01	.11	.51
			2	.06	-.06	.39	.59
			3	-.05	-.11	.58	.66
			4	-.12	-.14	.68	.72
			$\infty$		-.33	1.03	
0.2	0.6	0.8	1	2.96	.02	-4.41	.32
			2	1.81	-.12	-2.56	.55
			3	.98	-.24	-1.22	.74
			4	.43	-.34	-.35	.90
			$\infty$		-1.28		2.42
0.4	0.6	0.8	1	1.95	-.15	-1.44	.49
			2	1.41	-.20	-.94	.54
			3	.84	-.27	-.42	.60
			4	.39	-.35	0	.68
			$\infty$		-1.28		1.53

This completes the list of simple deductions from the relation Eq. (25). In general, any case can of course be calculated directly from this relation. In Table I we list, by way of example, some cases selected more or less at random giving for each case and for each mode the two values of  $Q$  (normalized to  $Ne\pi(c^2 - b^2)$ , the amount of charge per unit axial length in the electron cloud) between which there is instability. We also list the two corresponding values of the potential [normalized to  $Ne(c^2 - b^2)/2\epsilon_0$ ] between the inner and outer cylinders between which there is instability. The cases listed allow one to see the effect of varying each of the geometrical quantities  $a$ ,  $b$ ,  $c$ , and  $d$  in turn, holding the others constant. For  $l \rightarrow \infty$ , it can be seen from Eq. (25) that the two limiting values of  $Q$  converge after normalization to  $-b^2/(c^2 - b^2)$ ; this value and the corresponding limiting potential is also listed for each case.

#### COUETTE-FLOW ANALOGY AND AXIAL PERTURBATIONS

An exact analogy exists between the two-dimensional electromagnetic problem discussed in this paper, and the two-dimensional motion of an incompressible frictionless fluid, the velocity fields being the same in each case. The incompressibility of the fluid flow field is guaranteed by Eq. (5). In the electromagnetic case, the electron density  $n$  is related to the potential  $\phi$  by Poisson's equation

$$ne/\epsilon_0 = \nabla^2 \phi. \quad (33)$$

The conservation of charge then gives, from Eq. (7),

$$D(\nabla^2 \phi)/Dt = 0. \quad (34)$$

In the fluid case,  $\phi$  is related to the velocity components by Eq. (3) and Eq. (4), and therefore has the character of a stream function. The vorticity,  $\zeta$ , is then given by

$$\zeta = [\text{curl } \mathbf{v}]_z \propto \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi. \quad (35)$$

The vorticity of a fluid element is conserved, following the motion of a perfect incompressible fluid. Hence

$$D(\nabla^2 \phi)/Dt = 0. \quad (36)$$

Finally, at a solid boundary the normal component of velocity vanishes, corresponding exactly (through the relation  $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ ) to the vanishing of the tangential electric field at a perfect conductor.

The purpose of bringing out the above analogy is to be able to make use of the substantial body of

work<sup>11,12</sup> dealing with the stability of two-dimensional plane shear flows and flows between rotating cylinders. Indeed, reference has already been made to this work in connection with the problem of the continuous spectrum of eigenvalues. The former case we shall not discuss in this paper. For the latter case, a well-known result of Rayleigh<sup>13</sup> states that a rotating fluid is stable only if

$$\frac{d}{dr} (rv_0)^2 \geq 0. \quad (37)$$

This result is obtained from simple considerations of energy and angular momentum. The analogous electromagnetic condition would be

$$\frac{d}{dr} (rE_0)^2 \geq 0. \quad (38)$$

In regions 1 and 3 [Eqs. (15) and (17)] this condition is marginally fulfilled. In region 2 it reduces to

$$E_0 \leq 0. \quad (39)$$

This condition will be satisfied at  $r = b$  if

$$Q \leq 0. \quad (40)$$

We have in the foregoing seen that if  $Q$  is sufficiently large and positive, any geometry can be stabilized. What is the meaning of this apparent paradox?

The result [Eq. (37)] appears to be concerned only with conditions in the plane. In reality, however, it depends for its validity upon the possibility of an interchange which can take place only with motions in the axial direction. Formally then, at least, it is hardly surprising that an analysis neglecting motion in this direction should arrive at results which are quite different from Eq. (37). We still have the possibility, however, that any stability predicted on the basis of Eq. (25) in violation of Eq. (37) may be spurious since axial motion may in fact allow interchanges to take place.

We shall confine ourselves in this regard to a few observations. In the first place, when three-dimensional motions are considered, the analogy discussed breaks down. This is seen most simply as follows:  $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$  implies  $E_z = 0$  and hence no axial fields. But the equation of motion of the fluid is governed simply by the axial pressure gradient. To obtain axial motions in the electromagnetic problem, we are obliged to introduce more physics, and in particular we must write an equation governing the desired axial motion. Such an equation should

<sup>11</sup> S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Clarendon Press, Oxford, 1961).

<sup>12</sup> C. C. Lin, *The Theory of Hydrodynamic Stability* (Cambridge University Press, London, 1955).

<sup>13</sup> Lord Rayleigh, *Scientific Papers* (Cambridge University Press, London, 1920), Vol. VI, p. 447.



bring in the effects of finite electron "temperature" and mass. More correctly, one should use the Vlasov system of equations to obtain a kinetic description of the situation.

At present, effects of this type and their implications are not fully understood. It is clear, however, that a high electron temperature, corresponding to easy motion along field lines, will have a strong tendency to nullify electric fields in the  $z$ -direction, and hence to validate our two-dimensional results.

**CONCLUDING REMARKS**

It has been demonstrated that, when axial effects can be ignored, proper selection of dimensions and potentials can ensure stability against the diocotron effect in cylindrical geometries.

*Note added in proof.* Contrary to the statement

in the introduction, Buneman<sup>14</sup> has considered the diocotron instability in a cylindrical geometry. However, he treats the case  $\omega_p^2/\omega_c^2 = 1$  which is unstable in all geometries. Also, Feinstein and Kino<sup>15</sup> have used the adiabatic equation of motion (3), but in a somewhat different context.

**ACKNOWLEDGMENTS**

It is a pleasure to acknowledge helpful discussions with W. Knauer, G.S. Janes, and H.E. Petschek.

This work has been supported jointly by Headquarters, National Aeronautics and Space Administration, Office of Advanced Research and Technology, Washington, D. C., under Contract NAS w-1101, and the Air Force Office of Scientific Research of the Office of Aerospace Research, United States Air Force under Contract No. AF 49(638)-659.

<sup>14</sup> O. Buneman, *J. Electron.* **3**, 1 (1957).

<sup>15</sup> J. Feinstein and G. S. Kino, *Proc. I. R. E.* **45**, 10 (1957).

**Kelvin-Helmholtz Instability in an Anisotropic Plasma**

S. P. TALWAR\*

*Theoretical Division, Goddard Space Flight Center, National Aeronautics and Space Administration, Greenbelt, Maryland*

(Received 2 November 1964; final manuscript received 21 January 1965)

The instability arising due to a tangential discontinuity of velocity in an unbounded, anisotropic plasma subject to a uniform magnetic field is investigated using the Chew, Goldberger, and Low approximation. Formal conditions for monotonic instability and the growing-wave instability are written. It is found that monotonic instability is possible, for a range of relative speeds, only if the plasma is characterized by the condition  $2p_{\parallel} > p_{\perp}$ . For  $2p_{\parallel} < p_{\perp}$  the configuration is, however, overstable for relative speeds less than a certain critical value. The bearing of the results on the stability of the magnetospheric boundary and the coronal streamers is discussed and it is suggested that a possible fine structure in the solar wind may satisfy the requirement for instability and thus produce irregularities of plasma properties and the magnetic field in the medium.

**INTRODUCTION**

THE purpose of the present paper is to investigate the Kelvin-Helmholtz instability, arising because of a tangential discontinuity of velocities between two streams of a homogeneous, nondissipative, anisotropic plasma. This problem is of interest in a variety of astro- and geophysical situations, e.g., the stability of the magnetospheric boundary, coronal streamers moving through the solar wind. The instability of the magnetospheric boundary, if ascertained, would help in understanding of the phenomena of magnetic storms, aurorae,

and radiation belts. Again the solar wind may possess a fine structure resulting in a tangential discontinuity of velocity between adjacent streams. The instability of this fine structure would result in production of irregularities of plasma properties and the magnetic field in the incoming solar wind and this may be the cause of the irregularity in the direction and the magnitude of the solar wind magnetic field vector as given by Explorer 10 and 12, and IMP I measurements.<sup>1</sup>

Previous investigations on Kelvin-Helmholtz in-

<sup>1</sup> L. J. Cahill and P. G. Amazeen, *J. Geophys. Research* **68**, 1835 (1963); J. P. Heppner, N. F. Ness, C. S. Scarce, and T. L. Skillman, *ibid.* **68**, 1 (1963); N. F. Ness, C. S. Scarce, and J. B. Seek, *ibid.* **69**, 3531 (1963).

\* On leave from Department of Physics and Astrophysics, Delhi University, India.