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Nuclear diagnostics and Magnetic Resonance Imaging

Week 1; Lecture 1; Section 3: Nuclear decay, revision

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Section 3

Revision of theory of nuclear decay

Radioactive decay

Observation:

"Activity", $A =$ number of disintegrations per second, decays exponentialy with time

So, if there are N nuclei at time t:

$$N(t) = N(0)\exp(-\lambda t) \qquad \ldots \text{ and so } \ldots \qquad \frac{dN}{dt} = -\lambda N$$

λ is the decay constant. The "lifetime", τ, and "halflife", $t_{\frac{1}{2}}$, are related to λ by:

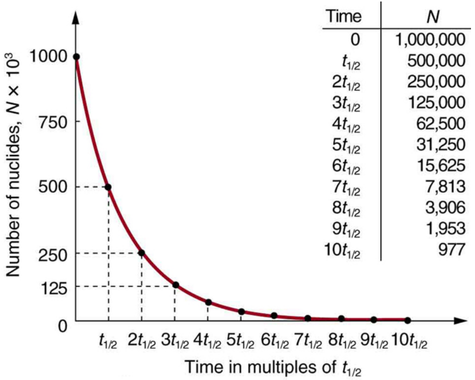
$$\tau = \frac{1}{\lambda} \qquad \ldots \text{ and } \ldots \qquad t_{\frac{1}{2}} = \frac{\ln(2)}{2}\tau = \frac{\ln(2)}{2}\frac{1}{\lambda}$$

The observation implies that the chance that a nucleus will decay per unit time is constant and is given by λ

Activity

Activity: $A = \left|\frac{dN}{dt}\right| = \gamma N$

- SI unity of activity: Becquerel (Bq):
  - $1\,\text{Bq} = 1$ decay per second
- Curie (Ci):
  - $1\,\text{Ci} = 3.7 \times 10^{10}\,\text{Bq}$



https://courses.lumenlearning.com/physics/chapter/31-5-half-life-and-activity/

Branching ratio

Decay constant, λ, determines the decay rate. The 'lifetime', τ is defined to be:

$$\tau = \frac{1}{\lambda} \qquad \ldots \text{ and so } \ldots \qquad \lambda = \frac{1}{\tau}$$

The decay rate for the transition of X into Y may be calculated using "Fermi's Golden Rule":

$$\lambda_{X \to Y} = \eta \left| M_{X \to Y} \right|^2 \rho_f$$

Where η is a constant and ρ_f is the density of final states. $M_{X \to Y}$ is the quantum-mechanical 'matrix element' for the transition $X \to Y$. Some radionuclides may decay via more than one route. For such nuclei:

$$\lambda_T = \lambda_{X \to Y} + \lambda_{X \to Z} + \cdots = \sum_i \lambda_i$$

λ_T is the 'total decay rate' (sometimes referred to as 'total width'). The λ_i are the 'partial' decay rates, or 'partial widths'.

Branching ratios; an additional constraint

'Branching ratio' (BR): fraction of all decays that result in a particular final state:

$$BR = \frac{\lambda_{X \to Y}}{\lambda_T}$$

Decay chain may include beneficial radiation, suitable for imaging, and harmful radiation.

Example: ^{131}I decay — $^{131}\mathrm{I}(e^-, \gamma)^{131}\mathrm{Xe}$

- Some γs in useful range for imaging, but
- e^- and low-energy γs simply deposit dose.

Has application in therapy, e.g., thyroid tumours.

Not widely used today for imaging.

So, consider ^{123}I, which decays via EC.

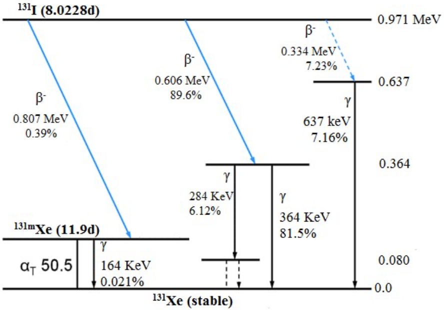
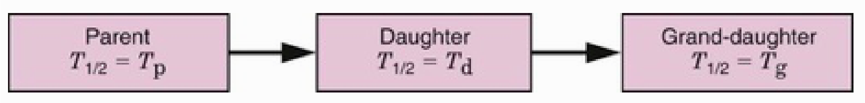


Figure from https://www.researchgate.net/publication/295919808_Radioiodine_I-131_for_Diagnosing_and_Treatment_of_Thyroid_Diseases

Parent-daughter decay chain



Branching ratio [Parent $\rightarrow$ Daughter] $= \beta$

Rate of ‘decay’of daughter nuclei:

$$\begin{aligned}\frac{dN_D}{dt} &= \lambda_P N_P \beta - \lambda_D N_D \\ &= \lambda_P N_{P0} \beta \exp\left(-\lambda_P t\right) - \lambda_D N_D\end{aligned}$$

I.e.:

$$\frac{dN_D}{dt} + \lambda_D N_D - \lambda_P \beta N_{P0} \exp\left(-\lambda_P t\right) = 0\,.$$

Solution:

$$N_D = \frac{\lambda_P}{\lambda_D - \lambda_P} \beta N_{P0} \left[\exp(-\lambda_P t) - \exp(-\lambda_D t)\right] + N_{D0} \exp(-\lambda_D t)$$

Or in terms of activation:

$$A_D = \frac{\lambda_D}{\lambda_D - \lambda_P} \beta A_{P0} \left[\exp(-\lambda_P t) - \exp(-\lambda_D t)\right] + A_{D0} \exp(-\lambda_D t) \qquad (1)$$

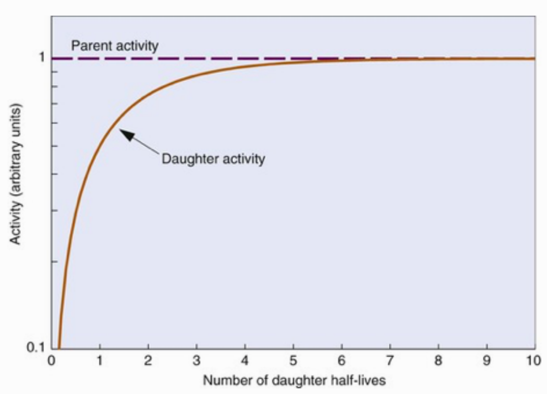
$T_P >> T_D$; secular equilibrium

$T_P >> T_D \Rightarrow \frac{\lambda_P}{\lambda_D} << 1$ and $\exp(-\lambda_P t) \sim 1$.

So, equation 1 becomes:

$$A_D = \beta A_{P0}\left[1 - \exp(-\lambda_D t)\right] + A_{D0}\exp(-\lambda_D t)$$

If $A_{D0} = 0$ and $\beta = 1$, then the build up of N_D reaches ‘secular equilibrium after 5–6 T_D.



$T_P > T_D$; transient equilibrium

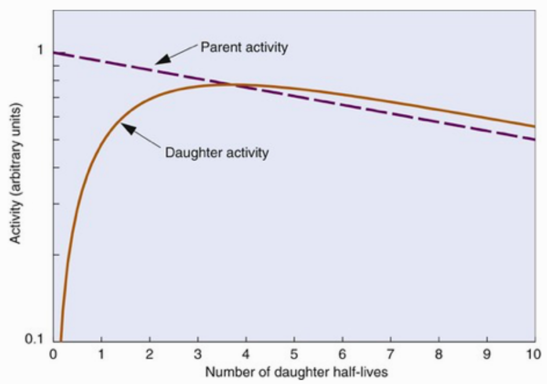
Transient equilibrium occurs at $t_{\rm eq}$ given by:

$$t_{\rm eq} = \frac{\ln\left[\frac{\lambda_P}{\lambda_D}\right]}{\lambda_P - \lambda_D}$$

At this time the activity of the daughter is a maximum, so one may write:

$$t_{\rm max} = t_{\rm eq} = \frac{1.44\, T_P T_D}{T_P - T_D} \ln\left[\frac{T_P}{T_D}\right]$$

If $A_{D0} = 0$, $\beta = 1$, and $T_D = 0.1\, T_P$, then build up and decay of N_D reaches 'transient equilibrium' after $\sim 2.6\, T_D$.



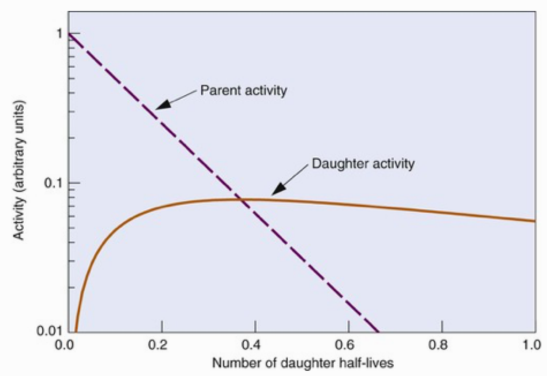
$T_P < T_D$; no equilibrium

Maximum activity of daughter is still given by:

$$t_{\rm max} = t_{\rm eq} = \frac{1.44\, T_P\, T_D}{T_P - T_D} \ln \left[\frac{T_P}{T_D} \right]$$

The daughter activity grows until $t_{\rm max}$ and then decreases. The parent activity 'falls away' and therefore fails to replenish the daughter.

If $A_{D0} = 0$, $\beta = 1$, and $T_D = 10\, T_P$, then build up of N_D reaches maximum activity at $\sim 0.26\, T_D$.



Summary of section 3

Radioactive decay law:

$$\frac{dN}{dt} = N(0)\exp\left(-\lambda t\right)$$

implies that chance that a nucleus will decay per unit time is λ

Branching ratio for decay $X \to Y$: $\mathrm{BR} = \frac{\lambda_{X\to Y}}{\lambda_T}$

Can solve for evolution of samples of parent, daughter, grand-daughter, etc. in decay chain and distinguish between:

- Secular equilibrium for $T_P >> T_D$
- Transient equilibrium for $T_P > T_D$
- No equilibrium for $T_P < T_D$