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Magnetic Resonance Imaging

Week 3; Lecture 7; Section 2: Quantum mechanical foundations of MRI

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Section 2

Quantum mechanical foundations

Theoretical description; a hybrid of quantum and classical

Nuclear magnetic resonance & MRI are both inherently quantum mechanical effects:

- Signal is generated by manipulating the *spins* of hydrogen nuclei:
  - Spin is postulated to explain hyperfine structure, Stern-Gerlach experiment, ...
  - Understood theoretically through the symmetries of space and time
- Magnetic moment of proton, $\boldsymbol{\mu}$, is related to the proton spin, $\mathbf{s}$, by:

$$\boldsymbol{\mu} = \gamma \mathbf{s}$$

where γ is the "gyromagnetic ratio"

Hybrid, quantum/classical treatment:

- Quantum mechanics: energy splitting and population in ground and excited state
- Classical: magnetisation vector, its precession, and the manipulation of the magnetisation vector to generate the signals used for imaging

Interaction of nuclear magnetic dipole with uniform magnetic field

The contribution, $\delta\mathcal{U}$, to the potential energy of a proton immersed in a magnetic field, **B**, is given by:

$$\delta\mathcal{U} = -\mathbf{B} \cdot \boldsymbol{\mu}$$

Lets consider a proton which, in the absence of a magnetic field has energy E. Applying the magnetic field introduces $\delta\mathcal{U}$ into the Schrödinger equation resulting in a splitting of the proton energy level such that $E \to E'$ given by:

$$E' = E \pm E_{m_s}$$

where

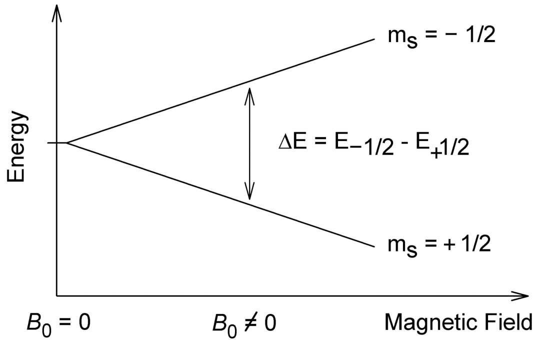
$$E_{m_s} = -m_S \gamma \hbar B_0$$

where m_s is the quantum number associated with the component of the proton spin parallel to **B**, $\hbar$ is Planck's constant divided by 2π, and B_0 is the magnitude of **B**

For the proton:

$$m_s = \pm\frac{1}{2}$$

Larmor equation



ΔE, splitting between two levels with $m_s = \pm\frac{1}{2}$:

$$\Delta E = \gamma \hbar B_0$$

Planck's law relates energy splitting to the angular frequency, ω, of the radiation required to excite the transition, therefore:

$$\Delta E = \hbar\omega$$

Writing ω in terms of γ and B_0 yields the Larmor equation:

$$\omega = \gamma B_0$$

Gyromagnetic ratios of some nuclei

Definition of gyromagnetic ratio, γ:

The gyromagnetic ratio, γ, of a particle or system is the ratio of its magnetic dipole moment to its angular momentum

For charged body of charge q, mass m rotating about an axis of symmetry:

$$\gamma = \frac{qe}{2m}$$

where e is the magnitude of the charge on the electron

For proton, $q = 1$, $m = m_p$, the proton mass.

$\gamma\!\!\!-$ is sometimes used instead of γ:

$$\gamma\!\!\!- = \frac{\gamma}{2\pi}$$

nucleus	γ (rad MHz T^{-1})	ɣ = γ / 2 π
^{1}H	267.513	42.576
^{2}H	41.065	6.536
^{3}He	203.789	32.434
^{7}Li	103.962	16.546
^{13}C	67.262	10.705
^{14}N	19.331	3.077
^{15}N	27.116	-4.316
^{17}O	36.264	5.772
^{19}F	251.662	40.053
^{23}Na	70.761	11.262
^{27}Al	69.763	11.103
^{31}P	108.291	17.235
^{57}Fe	8.681	1.382
^{63}Cu	71.118	11.319
^{67}Zn	16.767	2.669
^{129}Xe	73.997	11.777

Examples

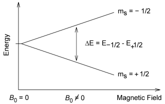
Larmor equation: $\omega = \gamma B_0 \quad \Rightarrow \quad \nu = \bar{\gamma} B_0$
For hydrogen nucleus, ^{1}H, $\bar{\gamma} = 42.58\,\mathrm{MHz/T}$

What is the resonance frequency for ^{1}H when:

- $B_0 = 1.5\,\mathrm{T}$?
- $B_0 = 3.0\,\mathrm{T}$?

What are the corresponding values for the energy splittings $\Delta E = h\nu$, where h is Planck's constant?

Populations in the two spin states



^{1}H in tissue in thermal equilibrium, so, partition between the populations in the two spin states follows the Boltzmann distribution:

$$\frac{N_+}{N_-} = \exp\left(-\frac{\Delta E}{k_{\mathrm{B}} T}\right)$$

where N_+ and N_- are the number of ^{1}H in $+\Delta E$ and $-\Delta E$ states respectively, k_{B} is Boltzmann's constant, and T is the temperature
For the human body, $k_{\mathrm{B}} T \approx 25.7\,\mathrm{meV}$, so:

$$\Delta E << k_{\mathrm{B}} T$$

Therefore, expanding the exponential and rearranging:

$$N_- - N_+ \approx N_S \frac{\Delta E}{2 k_{\mathrm{B}} T}$$

Magnetisation

Substituting for ΔE

$$N_- - N_+ \approx N_S \frac{\Delta E}{2 k_\mathrm{B} T} = N_S \frac{\gamma h B_0}{4\pi k_\mathrm{B} T}$$

For $B_0 = 1.5\,\mathrm{T}$:

$$\begin{aligned} \frac{N_- - N_+}{N_S} &\approx \frac{42.58 \times 10^6 \times 6.6 \times 10^{-34} \times 1.5}{2 \times 1.38 \times 10^{-23} \times 300} \\ &\approx 4.5 \times 10^{-6} \end{aligned}$$

i.e. only 4.5 in a million protons in the body are available for activation in MRI at $B_0 = 1.5\,\mathrm{T}$

Bulk magnetisation is measurable

Population-density "mismatch" of $\approx 3\,$ppm per Tesla arises due to fact that energy splitting is small compared to $k_{\mathrm{B}}T$

Bulk magnetisation still measurable because 1 gram of water contains 10^{22} ^{1}H

Summary of section 2

Larmor frequency, ω determined by the gyromagnetic ratio, γ, and the applied magnetic field, B_0: $\omega = \gamma B_0$

In presence of B_0 at temperature T the equilibrium magnetisation of a sample of hydrogen nuclei is small, but measurable, and aligned with the applied magnetic field