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Physics of Medical Imaging and Radiotherapy

Magnetic Resonance Imaging

Lecture 1; Section 2: Quantum mechanical foundations of MRI

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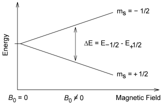
Section 2

Quantum mechanical foundations

Populations in the two spin states



^{1}H in tissue in thermal equilibrium, so, partition between the populations in the two spin states follows the Boltzmann distribution:

$$\frac{N_+}{N_-} = \exp\left(-\frac{\Delta E}{k_\mathrm{B} T}\right)$$

where N_+ and N_- are the number of ^{1}H in $+\Delta E$ and $-\Delta E$ states respectively, k_B is Boltzmann's constant, and T is the temperature
For the human body, $k_\mathrm{B} T \approx 25.7\,\mathrm{meV}$, so:

$$\Delta E << k_\mathrm{B} T$$

Therefore, expanding the exponential and rearranging:

$$N_- - N_+ \approx N_S \frac{\Delta E}{2 k_\mathrm{B} T}$$

Magnetisation

Substituting for ΔE

$$N_{-} - N_{+} \approx N_S \frac{\Delta E}{2 k_{\mathrm{B}} T} = N_S \frac{\gamma h B_0}{4\pi k_{\mathrm{B}} T}$$

For $B_0 = 1.5\,\mathrm{T}$:

$$\begin{aligned} \frac{N_{-} - N_{+}}{N_S} &\approx \frac{42.58 \times 10^6 \times 6.6 \times 10^{-34} \times 1.5}{2 \times 1.38 \times 10^{-23} \times 300} \\ &\approx 4.5 \times 10^{-6} \end{aligned}$$

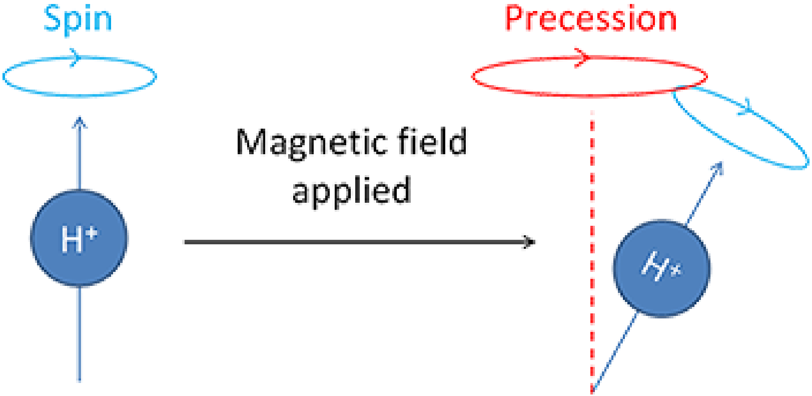
i.e. only 4.5 in a million protons in the body are available for activation in MRI at $B_0 = 1.5\,\mathrm{T}$

Bulk magnetisation is measurable

Population-density "mismatch" of $\approx 3\,\mathrm{ppm}$ per Tesla arises due to fact that energy splitting is small compared to $k_{\mathrm{B}}T$

Bulk magnetisation still measurable because 1 gram of water contains 10^{22} ^{1}H

Classical magnetic moment in magnetic field

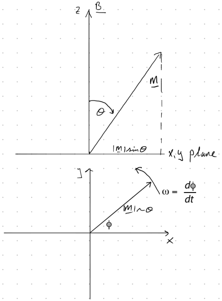


Magnetic moment that makes an angle with a magnetic field will precess around the magnetic-field axis.

Classical derivation of the Larmor equation

Classically, a magnetic moment, **M**, in a magnetic field **B**, experiences a torque given by the Bloch equation:

$$\frac{d\mathbf{M}}{dt} = \gamma\left(\mathbf{M} \times \mathbf{B}\right)$$



In time δt precession of $\underline{M}$ causes change in projection:

$$\underline{\delta M} = |\underline{M}|\sin\theta\,\omega\,\delta t\ \hat{\underline{\omega}}$$

M makes an angle θ w.r.t. **B**. So:

$$\mathbf{M} \times \mathbf{B} = \left(MB_0 \sin\theta\right)\hat{\boldsymbol{\omega}}$$

So:

$$\frac{d\mathbf{M}}{dt} = \left(\gamma MB_0 \sin\theta\right)\hat{\boldsymbol{\omega}} = \left(M\omega\sin\theta\right)\hat{\boldsymbol{\omega}}$$

Which gives the Larmor equation:

$$\omega = \gamma B_0$$

Examples

Larmor equation: $\omega = \gamma B_0 \quad \Rightarrow \quad \nu = \bar{\gamma} B_0$

Energy splitting: $\Delta E = \hbar\omega \quad \Rightarrow \quad \Delta E = h\nu$

$h = 4.1357 \times 10^{-15}\,\mathrm{eV\,s}$

For hydrogen nucleus, $^1\mathrm{H}$, $\bar{\gamma} = 42.58\,\mathrm{MHz/T}$

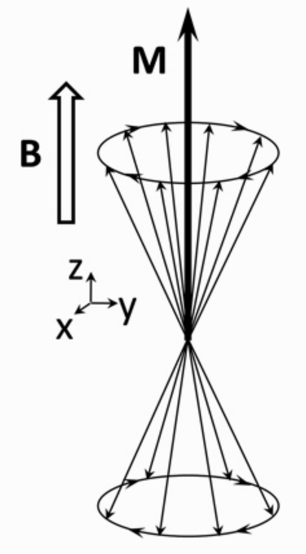
Calculating the values of ν and ΔE yields:

Magnetic field B_0 (T)	**Larmor frequency (MHz)**	**ΔE (eV)**
1.5	63.87	2.64E-07
3.0	127.74	5.28E-07

For comparison:

- FM radio waveband runs from 88.1 MHz to 108.1 MHz;
- $k_{\mathrm{B}} T = 2.59 \times 10^{-2}\,\mathrm{eV}$

Larmor precession



Ensemble of ^{1}H nuclei, the majority (by $\approx 3\,\text{ppm}\,\text{T}^{-1}$) orientated parallel to **B** precess at equilibrium around **B** at the Larmor angular frequency ω

Net magnetisation, **M**, produced is parallel to **B**.

There is no net magnetisation in the transverse (x, y) plane; sum of all contributions cancel

Result is that there is no change in the magnitude or direction of the magnetisation vector so no RF signal is produced

Key feature of MRI: manipulate **M** so as to produce a measurable RF signal

Summary of section 2

In presence of B_0 at temperature T the equilibrium magnetisation of a sample of hydrogen nuclei is small, but measurable, and aligned with the applied magnetic field

Magnetisation vector, **M**, created by unequal number of ^{1}H spins parallel and anti-parallel to the applied magnetic field **B**

The magnetisation vector precesses around the direction defined by the applied magnetic field at the Larmor frequency, ω

The Larmor frequency is given by:

$$\omega = \gamma B_0$$

This is the same Larmor frequency that was obtained in the quantum-mechanical discussion of the splitting of the energy level of the ^{1}H nucleus