January 1, 2022

Physics of Medical Imaging and Radiotherapy

Magnetic Resonance Imaging

Lecture 1; Introduction to MRI and quantum-mechanical foundations

K. Long (k.long@imperial.ac.uk)
Department of Physics, Imperial College London/STFC

Contents

Section 1

Introduction to MRI

'Guilt-free' imaging

Whole-body imager, Star Trek style

Nuclear diagnostics and X-ray imaging:

- Image constructed using ionising radiation
- Necessarily delivers dose to patient
- Dose implies risk of initiating disease

Magnetic resonance imaging (MRI):

- Image generated by exploiting magnetic moment of H nuclei
- Patient immersed in magnetic field
- No permanent harmful effects reported

Nuclear magnetic moment

Proton (and neutron) magnetic moment:

- Nucleons each have spin $\frac{1}{2}$
- Magnetic moment generated by nuclear charge
  - Contributions to nuclear spin arise from quarks and gluons
  - Quantitative explanation of nuclear magnetic moment is active field of research
- Magnetic moment, $\boldsymbol{\mu}$, is related to nuclear spin, $\mathbf{s}$ by:

$$\boldsymbol{\mu} = \gamma \mathbf{s}$$

where γ is the "gyromagnetic" ratio

Nuclear magnetic resonance

Effect of uniform magnetic field **B**:

- **B** provides "quantisation axis":
  $\Rightarrow$ nuclear dipoles align with magnetic field
- Proton spin is $\frac{1}{2}$, so only two states:
  Spin "up" and spin "down"
- Energy splitting; 2 energy levels:
  - Lower energy level has magnetic moment parallel to magnetic field
  - Higher energy level has magnetic moment anti-parallel to magnetic field
- Resonance:
  - Call energy splitting ΔE
  - Transitions between the two energy levels cause absorption or emission of electromagnetic (em) radiation for which $\Delta E = h\nu$
  - Resonance occurs when em radiation of frequency ν is injected

Magnetic resonance imaging

Magnetic resonance imaging (MRI) exploits this resonance

Steps:

- Apply uniform magnetic field, align proton (^{1}H) spins
- Apply radiation, at exactly ν, cause transitions between "spin up" & "spin down" states
- Turn off the radiation . . . and . . .
- "Listen" for radiation at exactly ν as the spins realign

Brilliant!
Simple principle and elegant technique, exploited in exquisitely sophisticated imaging systems.

Theoretical description; a hybrid of quantum and classical

Nuclear magnetic resonance & MRI are both inherently quantum mechanical effects:

- Signal is generated by manipulating the *spins* of hydrogen nuclei:
  - Spin is postulated to explain hyperfine structure, Stern-Gerlach experiment, ...
  - Understood theoretically through the symmetries of space and time
- Magnetic moment of proton, $\boldsymbol{\mu}$, is related to the proton spin, $\mathbf{s}$, by:

$$\boldsymbol{\mu} = \gamma \mathbf{s}$$

  where γ is the "gyromagnetic ratio"

Hybrid, quantum/classical treatment:

- Quantum mechanics: energy splitting and population in ground and excited state
- Classical: magnetisation vector, its precession, and the manipulation of the magnetisation vector to understand the signals used for imaging

Interaction of nuclear magnetic dipole with uniform magnetic field

The contribution, $\delta\mathcal{U}$, to the potential energy of a proton immersed in a magnetic field, **B**, is given by:

$$\delta\mathcal{U} = -\mathbf{B} \cdot \boldsymbol{\mu}$$

Lets consider a proton which, in the absence of a magnetic field, has energy E. Applying the magnetic field introduces $\delta\mathcal{U}$ into the Schrödinger equation resulting in a splitting of the proton energy level such that $E \rightarrow E'$ given by:

$$E' = E \pm E_{m_s}$$

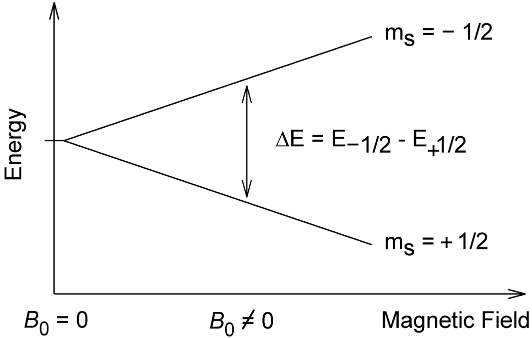
where

$$E_{m_s} = -m_s \gamma \hbar B_0$$

where m_s is the quantum number associated with the component of the proton spin parallel to **B**, $\hbar$ is Planck's constant divided by 2π, and B_0 is the magnitude of **B**
For the proton:

$$m_s = \pm\frac{1}{2}$$

Larmor equation



ΔE, splitting between two levels with $m_s = \pm\frac{1}{2}$:

$$\Delta E = \gamma \hbar B_0$$

Planck's law relates energy splitting to the angular frequency, ω, of the radiation required to excite the transition, therefore:

$$\Delta E = \hbar\omega$$

Writing ω in terms of γ and B_0 yields the Larmor equation:

$$\omega = \gamma B_0$$

Gyromagnetic ratios of some nuclei

Definition of gyromagnetic ratio, γ:

The gyromagnetic ratio, γ, of a particle or system is the ratio of its magnetic dipole moment to its angular momentum

For charged body of charge q, mass m rotating about an axis of symmetry:

$$\gamma = \frac{qe}{2m}$$

where e is the magnitude of the charge on the electron

For proton, $q = 1$, $m = m_p$, the proton mass.

$\gamma\!\!\!-$ is sometimes used instead of γ:

$$\gamma\!\!\!- = \frac{\gamma}{2\pi}$$

nucleus	γ (rad MHz T^{-1})	$\gamma\!\!\!-$ = γ / 2 π
^{1}H	267.513	42.576
^{2}H	41.065	6.536
^{3}He	203.789	32.434
^{7}Li	103.962	16.546
^{13}C	67.262	10.705
^{14}N	19.331	3.077
^{15}N	27.116	-4.316
^{17}O	36.264	5.772
^{19}F	251.662	40.053
^{23}Na	70.761	11.262
^{27}Al	69.763	11.103
^{31}P	108.291	17.235
^{57}Fe	8.681	1.382
^{63}Cu	71.118	11.319
^{67}Zn	16.767	2.669
^{129}Xe	73.997	11.777

Examples

Larmor equation: $\omega = \gamma B_0 \quad \Rightarrow \quad \nu = \bar{\gamma} B_0$
For hydrogen nucleus, ^{1}H, $\bar{\gamma} = 42.58\,\mathrm{MHz/T}$

What is the resonance frequency for ^{1}H when:

- $B_0 = 1.5\,\mathrm{T}$?
- $B_0 = 3.0\,\mathrm{T}$?

What are the corresponding values for the energy splittings $\Delta E = h\nu$, where h is Planck's constant?

Summary of section 1

Magnetic moment of proton exploited to provide energy splitting, ΔE, between spin-up and spin-down states in applied magnetic field

Injection of radio-frequency wave with a frequency that resonates with the splitting then used to manipulate population of protons in the spin-up and spin down states

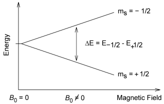
Images produced by manipulating applied magnetic field and frequency of RF field gradients

Larmor frequency, ω determined by the gyromagnetic ratio, γ, and the applied magnetic field, B_0: $\omega = \gamma B_0$

Section 2

Quantum mechanical foundations

Populations in the two spin states



^{1}H in tissue in thermal equilibrium, so, partition between the populations in the two spin states follows the Boltzmann distribution:

$$\frac{N_+}{N_-} = \exp\left(-\frac{\Delta E}{k_\mathrm{B} T}\right)$$

where N_+ and N_- are the number of ^{1}H in $+\Delta E$ and $-\Delta E$ states respectively, k_B is Boltzmann's constant, and T is the temperature
For the human body, $k_\mathrm{B} T \approx 25.7\,\mathrm{meV}$, so:

$$\Delta E << k_\mathrm{B} T$$

Therefore, expanding the exponential and rearranging:

$$N_- - N_+ \approx N_S \frac{\Delta E}{2 k_\mathrm{B} T}$$

Magnetisation

Substituting for ΔE

$$N_- - N_+ \approx N_S \frac{\Delta E}{2 k_\mathrm{B} T} = N_S \frac{\gamma h B_0}{4\pi k_\mathrm{B} T}$$

For $B_0 = 1.5\,\mathrm{T}$:

$$\begin{aligned} \frac{N_- - N_+}{N_S} &\approx \frac{42.58 \times 10^6 \times 6.6 \times 10^{-34} \times 1.5}{2 \times 1.38 \times 10^{-23} \times 300} \\ &\approx 4.5 \times 10^{-6} \end{aligned}$$

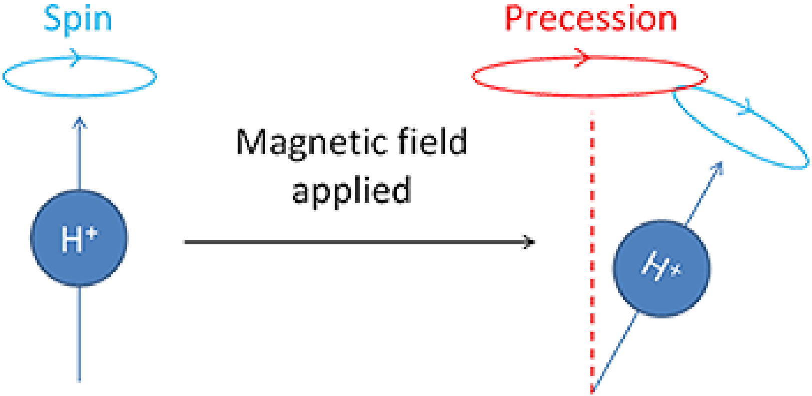
i.e. only 4.5 in a million protons in the body are available for activation in MRI at $B_0 = 1.5\,\mathrm{T}$

Bulk magnetisation is measurable

Population-density "mismatch" of $\approx 3\,$ppm per Tesla arises due to fact that energy splitting is small compared to $k_{\mathrm{B}}T$

Bulk magnetisation still measurable because 1 gram of water contains 10^{22} ^{1}H

Classical magnetic moment in magnetic field

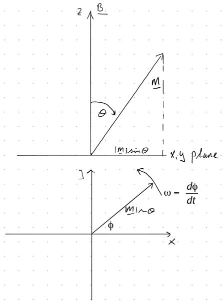


Magnetic moment that makes an angle with a magnetic field will precess around the magnetic-field axis.

Classical derivation of the Larmor equation

Classically, a magnetic moment, **M**, in a magnetic field **B**, experiences a torque given by the Bloch equation:

$$\frac{d\mathbf{M}}{dt} = \gamma\left(\mathbf{M} \times \mathbf{B}\right)$$



In time δt precession of $\underline{M}$ causes change in projection:

$\delta \underline{M} = |\underline{M}| \sin\theta\, \omega\, \delta t\, \hat{\underline{\omega}}$

M makes an angle θ w.r.t. **B**. So:

$$\mathbf{M} \times \mathbf{B} = \left(M B_0 \sin\theta\right)\hat{\boldsymbol{\omega}}$$

So:

$$\frac{d\mathbf{M}}{dt} = \left(\gamma M B_0 \sin\theta\right)\hat{\boldsymbol{\omega}} = \left(M\omega \sin\theta\right)\hat{\boldsymbol{\omega}}$$

Which gives the Larmor equation:

$$\omega = \gamma B_0$$

Examples

Larmor equation: $\omega = \gamma B_0 \quad \Rightarrow \quad \nu = \gamma\!\!\!-B_0$

Energy splitting: $\Delta E = \hbar\omega \quad \Rightarrow \quad \Delta E = h\nu$

$h = 4.1357 \times 10^{-15}\,\mathrm{eV\,s}$

For hydrogen nucleus, $^{1}\mathrm{H}$, $\gamma\!\!\!- = 42.58\,\mathrm{MHz/T}$

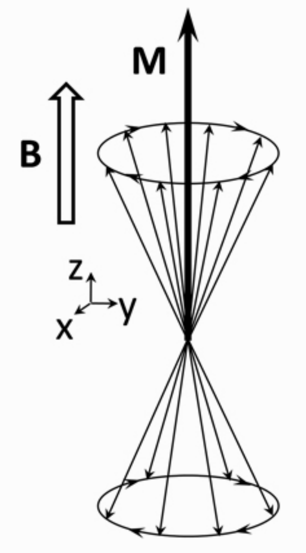
Calculating the values of ν and ΔE yields:

Magnetic field B_0 (T)	**Larmor frequency (MHz)**	**ΔE (eV)**
1.5	63.87	2.64E-07
3.0	127.74	5.28E-07

For comparison:

- FM radio waveband runs from 88.1 MHz to 108.1 MHz;
- $k_{\mathrm{B}} T = 2.59 \times 10^{-2}\,\mathrm{eV}$

Larmor precession



Ensemble of ^{1}H nuclei, the majority (by $\approx 3\,\mathrm{ppm\,T^{-1}}$) orientated parallel to **B** precess at equilibrium around **B** at the Larmor angular frequency ω

Net magnetisation, **M**, produced is parallel to **B**.

There is no net magnetisation in the transverse (x, y) plane; sum of all contributions cancel

Result is that there is no change in the magnitude or direction of the magnetisation vector so no RF signal is produced

Key feature of MRI: manipulate **M** so as to produce a measurable RF signal

Summary of section 2

In presence of B_0 at temperature T the equilibrium magnetisation of a sample of hydrogen nuclei is small, but measurable, and aligned with the applied magnetic field

Magnetisation vector, $\mathbf{M}$, created by unequal number of ^{1}H spins parallel and anti-parallel to the applied magnetic field $\mathbf{B}$

The magnetisation vector precesses around the direction defined by the applied magnetic field at the Larmor frequency, ω

The Larmor frequency is given by:

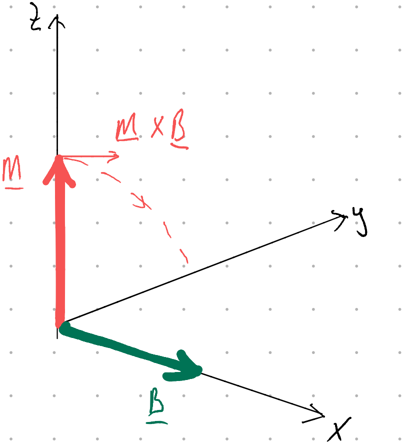
$$\omega = \gamma B_0$$

This is the same Larmor frequency that was obtained in the quantum-mechanical discussion of the splitting of the energy level of the ^{1}H nucleus

Section 3

Rotating the magnetisation

First, a static example



Consider magnetisation **M** parallel to z axis and **B** parallel to the x axis, as shown

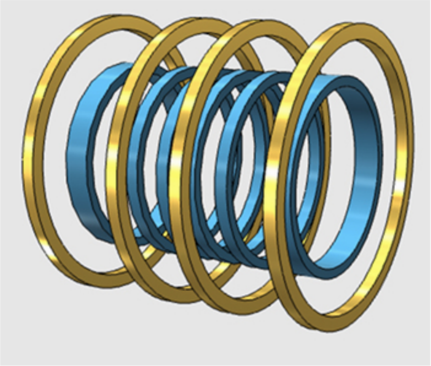
Torque, $\mathbf{M} \times \mathbf{B}$, is therefore parallel to the y axis

Net result is that **M** will precess around the x axis towards the y axis

This is what is done in MRI . . .

Rotating the magnetisation vector in MRI; principle

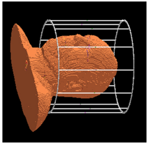
Main field, $\mathbf{B}_0$, produced with solenoid



Induces magnetisation **M** parallel to $\mathbf{B}_0$

To rotate **M** away from $\mathbf{B}_0$ require magnetic field in transverse (x, y) plane

Call the field in the x, y plane $\mathbf{B}_1$; can be produced with a variety of coil arrangements, e.g. dipole or, more efficient, a "bird cage"



To cause **M** to precess require that magnetic field oscillates at the Larmor frequency, ω.

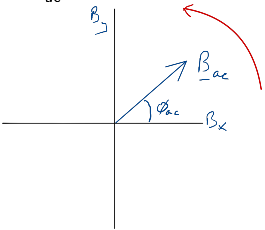
. . . i.e. require RF magnetic field $\mathbf{B}_1$

Rotating the magnetisation vector in MRI; mathematics

Take $\mathbf{B}_1$ to be "plane polarised" in x, y such that $B_{1_x} = B_1 \cos(\omega t + \alpha)$ and $B_{1_y} = B_1 \sin(\omega t + \beta)$; α and β are phases

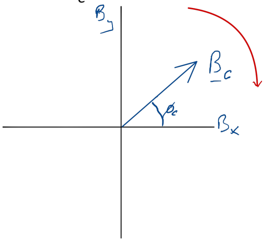
$\mathbf{B}_1$ can be rewritten in terms of two circularly polarised fields:

$\mathbf{B}_{1_{ac}}$; anti-clockwise



$$B_{1_{ac}} = \frac{B_1}{2};\ \phi_{ac} = \omega t + \alpha'$$

$\mathbf{B}_{1_c}$; clockwise



$$B_{1_c} = \frac{B_1}{2};\ \phi_c = \omega t + \beta'$$

Rotating the magnetisation vector in MRI

One of the two counter rotating fields will rotate in the same direction as the nuclear precession

Either $B_{1_{ac}}$ or B_{1_c} will appear stationary in the plane transverse to $\mathbf{B}_0$ in the frame that is co-rotating with the precession of the net magnetisation vector. Call the co-rotating field B_1^+

B_1^+ is equal to either $B_{1_{ac}}$ or B_{1_c} depending on the direction of $\mathbf{B}_0$

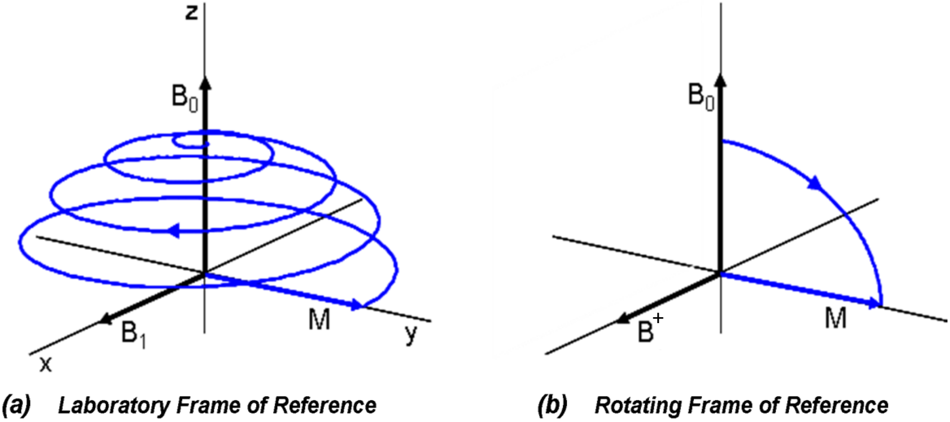
The field stationary in the rotating frame will therefore cause **M** to precess about a rotating axis in the transverse plane

The net result is that **M** can be rotated into the x, y plane where it will continue to precess

The precession of **M** in the x, y plane gives a detectable RF signal

Rotating the magnetisation vector in MRI

M is initially parallel to $\mathbf{B}_0$



The flip angle

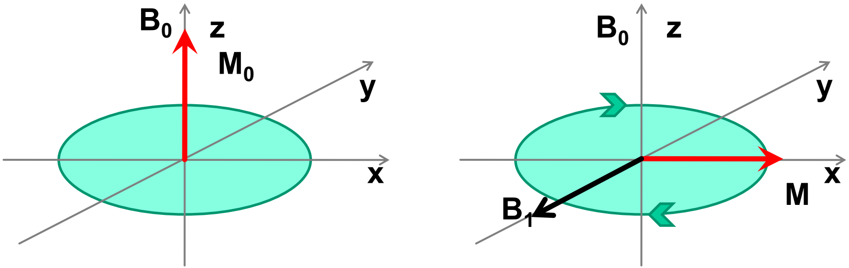
The flip angle, α, is proportional to the magnitude and duration of the RF pulse:

$$\alpha = \gamma B_1 t_P$$

where t_P is the duration of the RF pulse

90° pulse rotates magnetisation into transverse plane where it continues to precess

Effect of 90º RF Pulse



Example: calculating the duration of a 90° pulse

RF transverse magnetic field pulse is applied to rotate $\mathbf{M}$

The magnitude of B_1 is $10\,\mu$T (i.e. 10^{-5} T)

At what rate will $\mathbf{M}$ rotate away from the $\mathbf{B}_0$ axis?

How long will it take for the flip angle to reach 90°?

Example: calculating the duration of a 90° pulse

Half an answer . . .

The magnitude of B_1 is $10\,\mu$T (i.e. 10^{-5} T)

At what rate will **M** rotate away from the $\mathbf{B}_0$ axis?

It will rotate at the Larmor frequency, f_1, arising from the field B_1, i.e. $f_1 = \gamma\!\!\!-B_1$

How long will it take for the flip angle to reach 90°?

The angle can be obtained by solving the equation:

$$\frac{1}{4} = \gamma\!\!\!-B_1 t_P^{90^\circ} \text{ for } t_P^{90^\circ}$$

or $\frac{\pi}{2} = \gamma B_1 t_P^{90^\circ}$

Summary of section 3

Net magnetisation of ^{1}H spins caused to rotate using plane-polarised, time-varying magnetic field in the x, y plane

Precession of rotated net-magnetisation vector gives rise to RF signal which can be detected

Measurement of the RF signal from the precession of rotated net-magnetisation vector is the basis of MRI

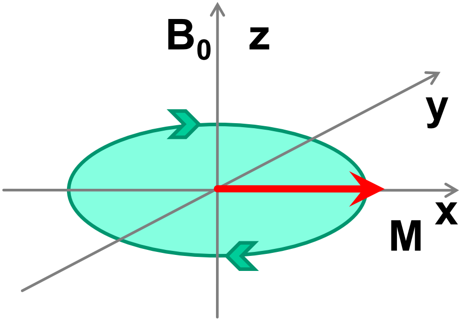
Section 4

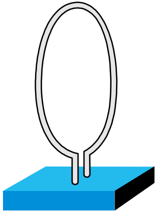
Free induction decay

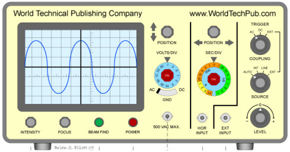
Detection of signal precession of magnetisation vector

M rotated using B_1 RF pulse. If flip angle α is not a multiple of 180°, then, result of B_1 pulse is a component of magnetisation in the x, y plane that is precessing

This yields an RF wave that can be detected

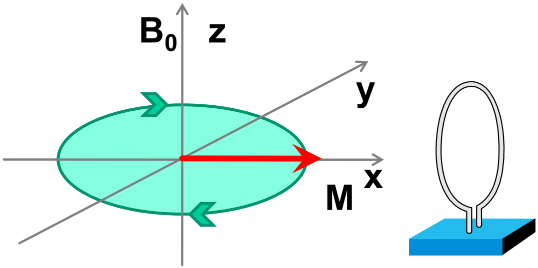


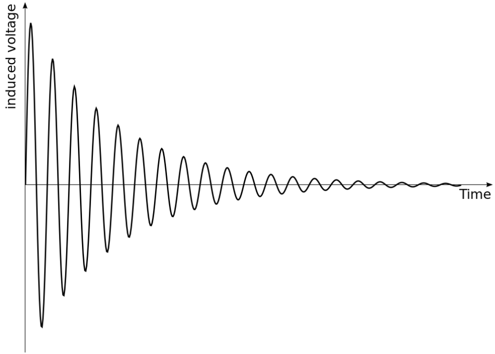




Free induction decay (FID)

Occurs when perturbing field (B_1) is turned off





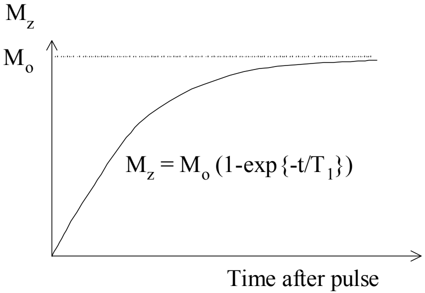
Note; exponential decay of amplitude of transverse magnetisation. Frequency of rotation remains the Larmor frequency corresponding to B_0

Spin-lattice (longitudinal) relaxation

When the B_1 pulse is turned off, the longitudinal magnetisation, M_z, recovers:

$$\frac{dM_z}{dt} = \frac{M_0 - M_z}{T_1} \qquad \Rightarrow M_z(t) = M_0 \left[1 - \exp\left(-\frac{t}{T_1}\right)\right]$$

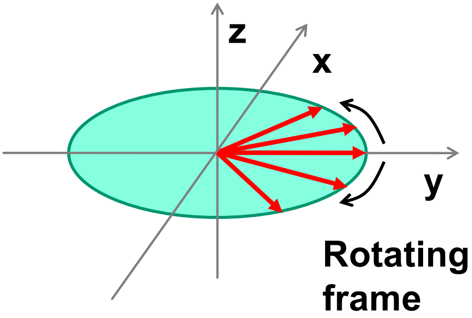
The process is characterised by a time constant T_1



Spin-lattice relaxation:

- ^{1}H spins relax to the low-energy state. Energy released returns to the "lattice" as heat
- Relatively ineffective thermal coupling to ^{1}H nuclei results in T_1 being large, typically $T_1 > 200$ ms

Spin-spin (transverse) relaxation



Contributions to M_{xy} smear out (decohere) rapidly

Causes M_{xy} to decay quickly

Some factors that affect the decoherence rate:

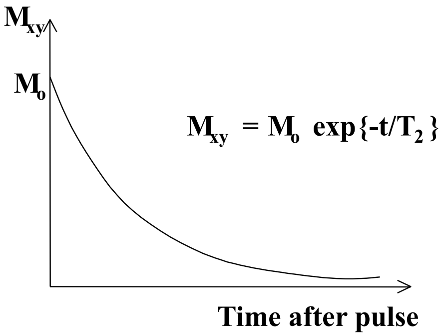
- Resonance frequency changes due to local magnetic fields
- Thermal excitations
- Spin "mobility"
- Presence of large molecules o, paramagnetic ions or molecules, outside interference

Spin-spin (transverse) relaxation

When the B_1 pulse is turned off, transverse magnetisation, decays:

$$\frac{dM_{xy}}{dt} = -\frac{M_{xy}}{T_2} \qquad \Rightarrow M_{xy}(t) = M_0 \exp\left(-\frac{t}{T_2}\right)$$

The process is characterised by a time constant T_2



Spin-spin relaxation:

- ^{1}H spins interact magnetically with their neighbours
- Coupling causes a variety of magnetic fields, causing a variety of precessions
- Effective randomisation of precessional modes leads to efficient depolarisation in transverse plane
- Results in T_2 being comparatively small, typically $T_2 \lesssim 100\,\text{ms}$

Relaxation times for a variety of tissues

Tissue Type	T1 (ms)	T2 (ms)
Adipose tissues	240-250	60-80
Whole blood (deoxygenated)	1350	50
Whole blood (oxygenated)	1350	200
Cerebrospinal fluid (similar to pure water)	4200 - 4500	2100-2300
Gray matter of cerebrum	920	100
White matter of cerebrum	780	90
Liver	490	40
Kidneys	650	60-75
Muscles	860-900	50

Relaxation times characteristic of tissue type

For materials important for human imaging
$T_1 > T_2$

Bloch equation revisited

Block equation may now be updated to include FID:

$$\frac{d\mathbf{M}}{dt} = \gamma \left(\mathbf{M} \times \mathbf{B}_0\right) - \frac{\mathbf{M}_{xy}}{T_2} + \frac{M_0 - M_z}{T_1}\hat{\mathbf{k}}$$

where:

- The first term describes the torque produced by the main (solenoid) field $\mathbf{B}_0$
- The second term describes the evolution of the transverse magnetisation vector $\mathbf{M}_{xy}$ due to the spin-spin interaction; time constant T_2
- The third term describes the evolution of the longitudinal magnetisation M_z due to the spin-lattice interaction; time constant T_1
- M_0 is the net magnetisation at equilibrium aligned with and proportional to $\mathbf{B}_0$

Complication: additional factors affecting the decay of the transverse magnetisation

T_2, the intrinsic spin-spin relaxation time is determined by non-reversible thermodynamic processes at the nuclear level.

The spin-spin relaxation time constant is reduced by a number of factors. A significant contribution comes from inhomogeneities in the main field $\mathbf{B}_0$

Inhomogeneities give rise to reversible thermodynamic processes. The associated relaxation of the transverse magnetisation is characterised by a time constant T_2'

The effective spin-spin relaxation time constant, T_2^* is given by:

$$\frac{1}{T_2^*} = \frac{1}{T_2} + \frac{1}{T_2'}$$

$T_2' < T_2$ and so $T_2^* < T_2$. Need to develop techniques to recover T_2 as this carries the clinically-relevant information

Comparison of T_2 and T_2'

T_2

- The individual dipoles that sum up to produce the transverse magnetization are not precessing at precisely the same rate
- As a water molecule tumbles due to thermal motions, each H nucleus feels a small, randomly varying magnetic field in addition to B_0
- When the random field adds to B_0, the dipole precesses a little faster, and when it subtracts from B_0, it precesses a little slower
- For each nucleus the pattern of random fields is different, so as time goes on the dipoles get progressively more out of phase with one another, and as a result no longer add coherently

T_2'

- The source of this T_2' effect is magnetic field inhomogeneity
- Because the precession frequency of the local transverse magnetization is proportional to the local magnetic field, any field inhomogeneity will lead to a range of precession rates
- Over time the precessing magnetization vectors will get out of phase with one another so that they no longer add coherently to form the net magnetization
- As a result, the net signal is reduced because of this destructive interference
- Static field offsets rather than fluctuating fields

Summary of section 4

Rotated net-magnetisation vector relaxes back to equilibrium orientation with time constant, T_1; spin-lattice relaxation time constant

Projection of net magnetisation vector in x, y plane decays away with time constant, T_2; spin-spin relaxation time constant

The effective spin-spin time constant, T_2^*, is a combination of the intrinsic spin-spin relaxation time constant (T_2) and the effect of "instrumental" effects such as inhomogeneities in the applied magnetic field (T_2')